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The m_2 product on chains (and not cochains)

Goal: We know that there are products on (linearized) Legendrian contact cohomology, which are collectively called <u>A_∞ products</u> or <u>Massey products</u>. Today we'll look at a single product structure, and pull it back to the level of chains (not cochains). This will be denoted by m_2.

Your Goal: Understand how the heights of Reeb chords change before and after m_2 multiplication. What effect does the augmentation have?

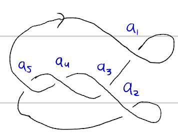
Setup: Let $\mathcal{A}$ denote the Chekanov-Eliashberg DGA of a particular Legendrian knot Λ having rotation number $r(\Lambda) = 0$.

Fix an augmentation $\varepsilon : (\mathcal{A}, \partial) \to (\mathbb{Z}_2, \text{zero differential})$. (In particular $\varepsilon(t) = 1$.) Let A denote the $\mathbb{Z}_2$ vector space having basis given by the Reeb chords of Λ. Define $\mathcal{A}^\varepsilon = (\mathcal{A} \otimes \mathbb{Z}_2)/(t=1)$, but we could start with computing $\mathcal{A}$ with $\mathbb{Z}_2$ coeffs from the outset.

We have seen that, as an algebra, $\mathcal{A}^\varepsilon = \bigoplus_{m \geq 0} A^{\otimes m}$, and recall that $A^{\otimes 0} = \mathbb{Z}_2$. Normally, to compute the linearized differential $\partial^\varepsilon_{(1)}$ we would ignore all terms of word length ≥ 2. Today we won't do that anymore.

Notation: We use subscripts inside of parentheses to denote grading from word length.

Recall: The augmentation defines a map $\Phi^\varepsilon : \mathcal{A}^\varepsilon \to \mathcal{A}^\varepsilon$ given by $\Phi^\varepsilon(a) = a + \varepsilon(a)$ and a differential $\partial^\varepsilon = \Phi^\varepsilon \circ \partial \circ (\Phi^\varepsilon)^{-1}$.



Example: We've done the right-handed trefoil before. Name the generators $a_1, \dots, a_5$ as in the figure. We computed $|a_1| = |a_2| = 1$. Let ε be the augmentation $\varepsilon(a_3) = 1$ and $\varepsilon = 0$ otherwise.

After a computation, $\partial^\varepsilon(a_1) = a_3 + a_5 + a_5 a_4 + a_5 a_4 a_3$

Break up the image as follows: $\underbrace{a_3 + a_5}_{\partial^\varepsilon_{(1)}(a_1)} + \underbrace{a_5 a_4}_{\partial^\varepsilon_{(2)}(a_1)} + \underbrace{a_5 a_4 a_3}_{\partial^\varepsilon_{(3)}(a_1)}$

and $\partial^\varepsilon(a_2) = a_3 + a_5 + a_4 a_5 + a_3 a_4 a_5$

Break up the image as follows: $\underbrace{a_3 + a_5}_{\partial^\varepsilon_{(1)}(a_2)} + \underbrace{a_4 a_5}_{\partial^\varepsilon_{(2)}(a_2)} + \underbrace{a_3 a_4 a_5}_{\partial^\varepsilon_{(3)}(a_2)}$

We see that the differential splits into a sum: $\partial^\varepsilon = \sum_n \partial^\varepsilon_{(n)}$

In particular, $\partial^\varepsilon_{(2)}(a_1) = a_5 a_4$ and $\partial^\varepsilon_{(2)}(a_2) = a_4 a_5$ and $\partial^\varepsilon_{(2)} = 0$ on all generators of grading zero (because $\partial^\varepsilon = 0$ on such). We note that $\partial^\varepsilon_{(2)} : \mathcal{A}^\varepsilon \to A^{\otimes 2}$. Furthermore, the degree of $\partial^\varepsilon_{(2)}$ with respect to the Maslov grading is -1.

Note: Every $\partial^\varepsilon_{(k)}$ $(k \geq 1)$ has degree -1 and maps $\mathcal{A}^\varepsilon \to A^{\otimes k}$, but each restricts to a vector space map $\partial_{(k)} : A \to A^{\otimes k}$. Each could be written as a matrix.

Definition: Let $m^\varepsilon_{(2)} = m_2 : (A^{\otimes 2})^* \to A^*$ denote the adjoint of the linear transformation $\partial^\varepsilon_{(2)}$.
We use the fact that A is finite-dimensional to identify $A^* \cong A$ using the same names for a basis for each. As a matrix m_2 is the transpose of $\partial^\varepsilon_{(2)}$.

Example: For the right-handed trefoil above, $m_2(a_5 \otimes a_4) = a_1$ and $m_2(a_4 \otimes a_5) = a_2$.

Note: How does m_2 relate to other product structures? There is a k-ary product $m_{(k)} : (A^{\otimes k})^* \to A^*$ given by taking the adjoint of $\partial_{(k)} : A \to A^{\otimes k}$. Notice that most $m_{(k)}$ will be the zero product. All $m_{(k)}$ have degree 1 with respect to Maslov grading.
Collectively, the $m_{(k)}$ satisfy the A_∞ relations: for each $\ell \geq 1$, $\boxed{0 = \sum_{i+j+k=\ell} m_{i+1+k} \circ (\mathbb{1}^{\otimes i} \otimes m_j \otimes \mathbb{1}^{\otimes k})}$
For instance, when $\ell = 2$, $\boxed{\delta \circ m_2(a \otimes b) = m_2(\delta a \otimes b) + m_2(a \otimes \delta b)}$
and recall that we use $\mathbb{Z}_2$ coefficients.

Exercise: Apply the adjoint to the A_∞ relations to get $\boxed{0 = \sum_{i+j+k=\ell} (\mathbb{1}^{\otimes i} \otimes \partial^\varepsilon_{(j)} \otimes \mathbb{1}^{\otimes k}) \circ \partial^\varepsilon_{(i+1+k)}}$
Check that this holds in the case $\ell = 1, 2, 3$
for the $\partial^\varepsilon_{(\ell)}$ defined for the right-handed trefoil. (Using $\mathbb{Z}_2$ coefficients!)

Note: The $m_{(k)}$ products induce an A_∞ product structure on linearized Legendrian contact cohomology which is denoted $\mu_{(k)}$. Here we investigate $m_{(2)}$ on chains.