Linearized Legendrian contact homology

We now have a hard algebraic problem before us: determine whether or not the DGAs $(\mathcal{A}_\Lambda, \partial_\Lambda)$ & $(\mathcal{A}_{\Lambda'}, \partial_{\Lambda'})$ are stable tame isomorphic.

Even obstructing quasi-isomorphism can be quite difficult, as the homology of these DGAs can have infinite dimension/rank in a fixed degree.

Today we'll discuss finite-dimensional approximations to $H_*(\mathcal{A}, \partial)$.

Linearizing an augmented DGA

Consider a DGA $(\mathcal{A}, \partial)$ over $\mathbb{Z}$, with

other ground rings okay too.

$$\mathcal{A} = \mathbb{Z}\langle a_1, a_2, \dots, a_n \rangle.$$

Let A be the graded $\mathbb{Z}$-module generated by $a_1, a_2, \dots, a_n$ — NOT an algebra. For any $k \geq 0$ we may define

$$(\mathcal{A})^k := \bigoplus_{m \geq k} A^{\otimes m}.$$

That is, $(\mathcal{A})^k \subseteq \mathcal{A}$ is the subalgebra generated by words of length at least k.

$$\mathcal{A} = (\mathcal{A})^0 \supsetneq (\mathcal{A})^1 \supsetneq \dots \supsetneq (\mathcal{A})^k \supsetneq \dots$$

Though $(\mathcal{A})^k$ is an algebra, it will not always inherit a differential from $\mathcal{A}$, because $\partial((\mathcal{A})^k)$ could include words of length $< k$.

Def. We call $(\mathcal{A}, \partial)$ augmented if the constant term in ∂a_i is 0, for $1 \le i \le n$.

Assuming that $(\mathcal{A}, \partial)$ is augmented, the differential $\partial: \mathcal{A} \to \mathcal{A}$ restricts to $\partial: (\mathcal{A})^k \to (\mathcal{A})^k$ because no word in $\mathcal{A}$ becomes shorter by applying ∂. It follows that $((\mathcal{A})^k, \partial)$ is a DGA.

Now notice that $(\mathcal{A})^2 \subsetneq (\mathcal{A})^1$, and ∂ induces a differential

$$\bar{\partial} : (\mathcal{A})^1/_{(\mathcal{A})^2} \longrightarrow (\mathcal{A})^1/_{(\mathcal{A})^2},$$

defined by $\bar{\partial}\left(w(\mathcal{A})^2\right) := (\pi_1 \circ \partial)(w)\,(\mathcal{A})^2$, where $\pi_1 : \mathcal{A} \to \mathcal{A}$ deletes words of length > 1.

In fact, $(\mathcal{A})^1/(\mathcal{A})^2$ is isomorphic to A* as a $\mathbb{Z}$-graded module.* So $(A, \bar{\partial})$ is a differential graded module.

Def. The linearized homology of an augmented DGA $(\mathcal{A}, \partial)$ is the homology of the resulting differential graded module $(A, \bar{\partial})$.

Augmenting $(\mathcal{A}_\Lambda, \partial_\Lambda)$

The Chekanov–Eliashberg DGA will not typically be augmented. However, we can sometimes conjugate the differential ∂_Λ by some automorphism of $\mathcal{A}_\Lambda$ to produce an augmented DGA.

Step ① Find an augmentation.

Fix a Legendrian Λ, along with an integer ρ which divides $2\mathrm{rot}(\Lambda)$. Then a ρ-graded augmentation of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ to S is a DGA chain map

$$\varepsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \longrightarrow (S, 0),$$

where $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is treated as a DGA with $\mathbb{Z}_\rho$ grading (induced by the $\mathbb{Z}$ grading).

We will focus on 0-graded augmentations, the existence of which imply that $r(\Lambda) = 0$. Let us also take $S = \mathbb{Z}_2$ for now.

Fact. Existence of augmentations is invariant under stable tame isomorphism.

Step ② Use ε to define an augmented DGA.

$$\mathcal{A}_\Lambda^\varepsilon := (\mathcal{A}_\Lambda \otimes \mathbb{Z}_2) / (t = \varepsilon(t)).$$

i.e., replace t in $\mathcal{A}_\Lambda$ with $\varepsilon(t)$.

This induces $\partial : \mathcal{A}_\Lambda^\varepsilon \longrightarrow \mathcal{A}_\Lambda^\varepsilon$, were ∂a_i is obtained from $\partial_\Lambda a_i$ via $t \mapsto \varepsilon(t)$.

We also have an automorphism

$$\phi^\varepsilon : \mathcal{A}_\Lambda^\varepsilon \longrightarrow \mathcal{A}_\Lambda^\varepsilon$$
$$a_i \longmapsto a_i + \varepsilon(a_i).$$

We obtain $\partial_\Lambda^\varepsilon$ by conjugating ∂ by ϕ^ε:

$$\partial_\Lambda^\varepsilon := \phi^\varepsilon \circ \partial \circ (\phi^\varepsilon)^{-1} : \mathcal{A}_\Lambda^\varepsilon \longrightarrow \mathcal{A}_\Lambda^\varepsilon.$$

Claim: $(\mathcal{A}_\Lambda^\varepsilon, \partial_\Lambda^\varepsilon)$ is augmented.

Indeed, $\partial_\Lambda^\varepsilon a_i = (\phi^\varepsilon \circ \partial \circ (\phi^\varepsilon)^{-1})\, a_i$ has constant term $(\varepsilon \circ \partial_\Lambda)\, a_i = 0$, $\forall\ 1 \le i \le n$.

Step ③. Linearize

Because $(\mathcal{A}_\Lambda^\varepsilon, \partial_\Lambda^\varepsilon)$ is augmented, we can use the above linearization procedure to obtain a differential graded module $(A_\Lambda^\varepsilon, \bar{\partial}_\Lambda^\varepsilon)$. We define the linearized Legendrian contact homology of Λ w.r.t. ε to be the homology of $(A_\Lambda^\varepsilon, \bar{\partial}_\Lambda^\varepsilon)$, denoted

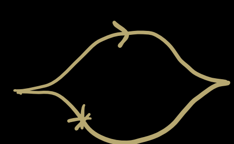
$$LCH_*^\varepsilon(\Lambda).$$

Problem! $LCH_*^\varepsilon(\Lambda)$ may depend on our choice of augmentation ε.

Thm (Chekanov) The collection

$$\left\{ LCH_*^\varepsilon(\Lambda) \;\middle|\; \begin{array}{l} \varepsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \to (S, 0) \\ \text{is an augmentation} \end{array} \right\}$$

is a Legendrian isotopy invariant.

Ex. $F(\Lambda) =$ 

Then $\Pi(\Lambda) =$ .

$$\mathcal{A}_\Lambda = \mathbb{Z}_2 \langle a, t^{\pm 1} \rangle, \quad |a| = 1, \quad |t| = |t^{-1}| = 0.$$

$$\partial_\Lambda a = 1 + t^{-1} \quad \longleftarrow \text{NOT augmented!}$$

An augmentation $\varepsilon : (A_\Lambda, \partial_\Lambda) \longrightarrow (\mathbb{Z}_2, 0)$ is determined by its effect on $1, a, t, t^{-1}$.

$|a| \neq 0 \Rightarrow \varepsilon(a) = 0.$

DGA map $\Rightarrow \varepsilon(1) = 1.$

$\varepsilon \circ \partial = 0 \Rightarrow \varepsilon(t^{-1}) = 1.$

$\therefore \varepsilon(t) = 1.$

Then $A_\Lambda^\varepsilon = \mathbb{Z}_2\langle a \rangle$, with $t \mapsto 1$.

$$\partial : A_\Lambda^\varepsilon \longrightarrow A_\Lambda^\varepsilon$$
$$a \mapsto 1 + \varepsilon(t^{-1}) = 0.$$

So $\partial = 0 \Rightarrow \partial_\Lambda^\varepsilon = 0.$

$A_\Lambda^\varepsilon = \mathbb{Z}_2\langle a \rangle$ as a *module*

$\partial_\Lambda^\varepsilon = 0.$

So $LCH_k^\varepsilon(\Lambda) = \begin{cases} \mathbb{Z}_2\langle a \rangle, & k = 1 \\ 0, & k \neq 1. \end{cases}$

$P^\varepsilon(z) = z.$

Linearized contact cohomology

Linearizing tossed out the product structure on LCH, but a product structure (+ more) can be defined on the linearized cohomology.

Let $(A_\Lambda^\varepsilon, \partial_\Lambda^\varepsilon)$ be as above. Then we can define $A_\varepsilon^\wedge := \mathrm{Hom}(A_\Lambda^\varepsilon, \mathbb{Z}_2)$, with generators $\check{a}_1, \ldots, \check{a}_n$.

Declare that $|\check{a}_i| = |a_i| + 1$ and define $\delta_\varepsilon^\wedge : A_\varepsilon^\wedge \to A_\varepsilon^\wedge$ by dualizing $\delta_\Lambda^\varepsilon$. Then

$$LCH^*_\varepsilon(\Lambda) := \text{homology of } (\hat{A}_\varepsilon, \hat{\delta}_\varepsilon).$$

As modules, $LCH^*_\varepsilon(\Lambda)$ tells us the same thing as $LCH^\varepsilon_*(\Lambda)$, but the product structure on the former has been used to distinguish otherwise indistinguishable Legendrians.