Legendrian Knots in $\mathbb{R}^3$

§1 Legendrian knots & isotopies

Consider the ODE

$$t^2 + (f'(t))^2 + (f(t))^2 = 1. \qquad (\star)$$

Very nonlinear and difficult. We can do a strange thing and identify its algebraic and analytic aspects:

$$\text{algebraic}: \quad x^2 + y^2 + z^2 = 1$$
$$\text{analytic}: \quad x = t,\ y = f'(t),\ z = f(t).$$

i.e., we imagine the ODE as an equation in the variables x, y, z, producing a surface in $\mathbb{R}^3$. We can then separately impose conditions on how $x, y,$ & z relate to one another.

So we can study ODEs as solutions to equations in $\mathbb{R}^3_{x,y,z}$.

Given a sol'n to $(\star)$, the embedding
$$t \mapsto (t, f'(t), f(t))$$
parametrizes a curve on $x^2 + y^2 + z^2 = 1$.

More generally, any ODE of order 1 has the form
$$G(t, f'(t), f(t)) = 0,$$
and solutions to this ODE correspond to curves

$t \mapsto (x(t), y(t), z(t))$

Satisfying (1) $G(x(t), y(t), z(t)) = 0$;

(2) $y(t) = \dfrac{z'(t)}{x'(t)}$. allows for reparametrization.

So we think of parametrized curves

$$t \mapsto (x(t), y(t), z(t))$$

which satisfy $z'(t) - y(t)\,x'(t) = 0$ as solutions to first-order ODEs. Such curves are called <u>Legendrian curves</u> in $\mathbb{R}^3$.

We are particularly interested in <u>Legendrian knots</u>

$$\gamma : S^1 \longrightarrow \mathbb{R}^3.$$

In fact, we'll immediately forget all this stuff about ODEs and attempt to classify Legendrian knots in $\mathbb{R}^3$ up to <u>Legendrian isotopy</u>.

Call $\gamma_0, \gamma_1 : S^1 \to \mathbb{R}^3$ <u>Legendrian isotopic</u> if there is a smooth family of parametrizations $\gamma_s : S^1 \to \mathbb{R}^3$ s.t. γ_0, γ_1 are what we want and each γ_s is Legendrian. i.e., $z_s'(t) - y_s(t)\,x_s'(t) = 0$.

We will (usually) consider two Legendrian knots to be equivalent iff they are Legendrian isotopic.

Notice that Legendrian isotopic knots must be smoothly isotopic. We'll see shortly that the converse is not true.

§2 Projections

Throughout, let's denote by Λ (the image of) a Legendrian knot in $\mathbb{R}^3$, with parametrization

$$t \mapsto (x(t), y(t), z(t)).$$

Knots are usually studied via <u>knot diagrams</u> in $\mathbb{R}^2$, obtained by projecting $\mathbb{R}^3 \to \mathbb{R}^2$ in some way. e.g.,

Because our knots are required to satisfy some geometric conditions, we have to be careful in how we project.

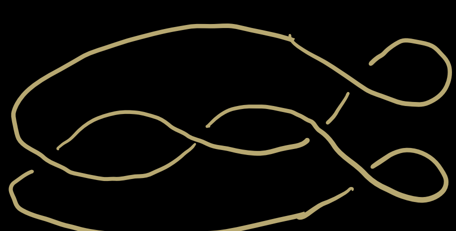
§2.1 Lagrangian projection

Consider the projection

$$\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x, y, z) \mapsto (x, y).$$

The Lagrangian projection of Λ is $\pi(\Lambda)$.

e.g.,



Given $\pi(\Lambda)$, we can recover Λ *up to Legendrian isotopy*. To see this, notice that

$\pi(\Lambda)$ is parametrized by $t \mapsto (x(t), y(t))$, and $z(t)$ satisfies $z'(t) = y(t)\, x'(t)$.

So we can deduce that

$$z(t) = z(0) + \int_0^t y(u)\, x'(u)\, du,$$

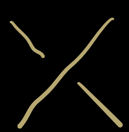
but we still don't know $z(0)$, so we can't recover Λ exactly. However,

$$(x(t), y(t), z(t) - s \cdot z(0)), \quad s \in [0,1]$$

is a Legendrian isotopy.

Generically, the only singular points of $\pi(\Lambda)$ are transverse double points. i.e., as long as we're willing to wiggle $\Lambda \subset \mathbb{R}^3$ a little via a small Leg. isotopy, we might see

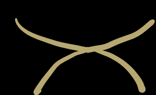
OR



in $\pi(\Lambda)$, but we'll never see

OR

OR



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<u>Drawback</u>: Not every knot diagram in $\mathbb{R}^2_{x,y}$ can be lifted. For instance, notice that

$$\int_0^1 y(t)\, x'(t)\, dt = z(1) - z(0) = 0$$

for $\pi(\Lambda)$. It's easy to draw knot diagrams where $\int_0^1 y(t)\, x'(t)\, dt \neq 0$. Other obstructions also exist.

Moreover, suppose we have $a<b$, with $(\pi\circ\gamma)(a)=(\pi\circ\gamma)(b)$. i.e., we have a double point of $\pi(\Lambda)$. Then

$$z(t) = z(0) + \int_0^t y(t)\,x'(t)\,dt$$

$$\begin{aligned} \Rightarrow z(b) - z(a) &= \int_a^b y(t)\,x'(t)\,dt \\ &= \int_{\gamma_{a,b}} y\,dx \\ &= \iint_R -1\,dA = -\operatorname{area}(R), \end{aligned}$$

where $\gamma_{a,b}$ is the portion of $t \mapsto (x(t), y(t))$ with $a \le t \le b$, and R is the region bounded by $\gamma_{a,b}$. So under/over crossing data is determined by the sign of $\displaystyle\int_a^b y(t)\,x'(t)\,dt$,

or equivalently by the orientation of $\gamma_{a,b}$.

R on right $\rightarrow \operatorname{area}(R) < 0$

$\Rightarrow z(b) - z(a) > 0$ ✓

R on left $\rightarrow \operatorname{area}(R) > 0$

$\Rightarrow z(b) - z(a) < 0$ ✓

R on right $\rightarrow \operatorname{area}(R) < 0$

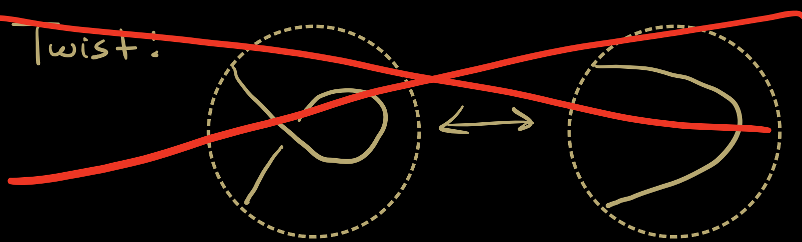
$\Rightarrow z(b) - z(a) > 0$ ✗

R on left $\rightarrow \operatorname{area}(R) > 0$

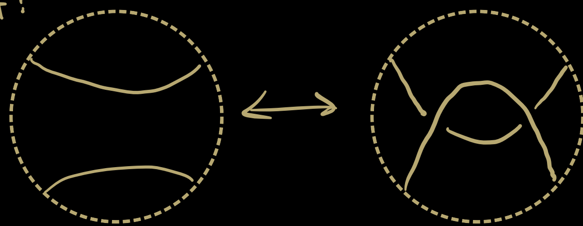
$\Rightarrow z(b) - z(a) < 0$ ✗

One consequence of this drawback is that we have to throw out a Reidemeister move:

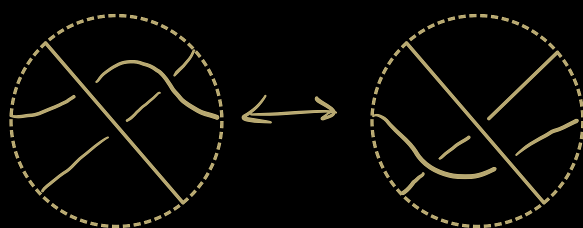
~~I. Twist:~~



II. Double point:



III. Triple point:



These moves can be arbitrarily reflected and rotated.

<u>Fact</u>: Any Legendrian isotopy can be realized by a sequence of R. moves of types II & III.

§2.2 Front projection

We'll define our primary invariant of Legendrian knots using the Lagrangian projection, but we have another projection which is often nicer.

Consider $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x,y,z) \mapsto (x,z).$$

We call $F(\Lambda)$ the front projection of Λ.
We can recover Λ from $F(\Lambda)$ on the nose.

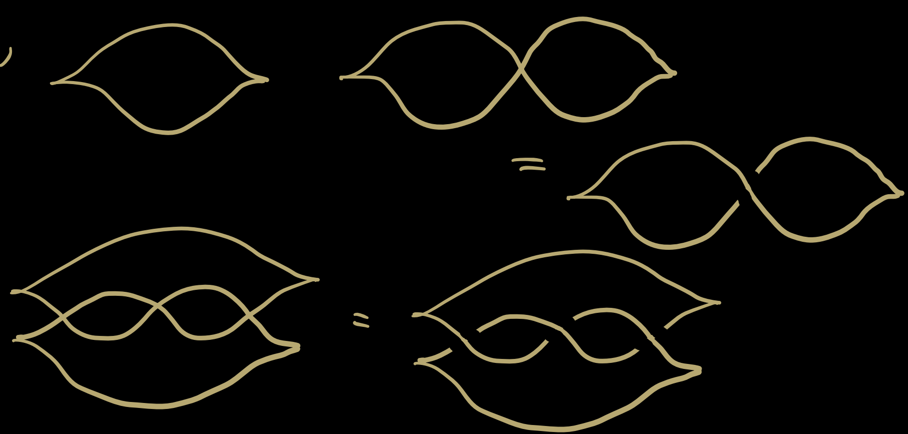
$$z'(t) - y(t)\, x'(t) = 0$$
$$\Rightarrow y(t) = z'(t) / x'(t).$$

No integration needed!

Upshot: We don't need to draw crossing data, b/c y-value is given by slope of strand.

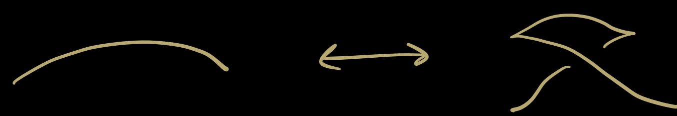
Note: Since $y(t)$ is finite, $x'(t)$ cannot vanish unless $z'(t)$ does as well. So no vertical tangencies.

e.g.,

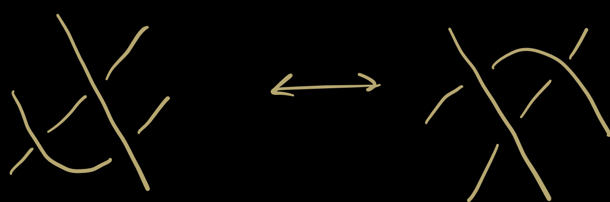


Front projections have the advantage that any diagram in $\mathbb{R}^2_{x,z}$ which is an immersion away from semicubical cusps and has no vertical tangencies will lift to a Legendrian knot.

We have Reidemeister moves for $F(\Lambda)$:

RI: 

RII: 

RIII: 

We have to be more careful w/ reflections and rotations now...

§2.3 Translating

Lemma (Ng, 2003)

Given $F(\Lambda)$, one can produce a diagram which is planar isotopic to $\pi(\Lambda')$, where Λ' is Leg. isotopic to Λ, via the following resolutions:

Moral: The Lagrangian projection is good for defining our invariant, but Legendrian isotopies are easier in the front projection.

§3 Classical invariants

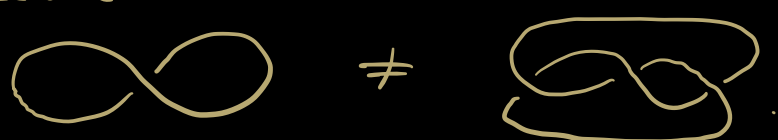
In general, the question of whether two Legendrian knots are Legendrian isotopic is a hard one. The standard strategy is to produce invariants — quantities which are not changed by Legendrian isotopy.

① Smooth knot type

Every Legendrian isotopy is a smooth isotopy, so Legendrian isotopic knots must have the same smooth knot type. Notation: $k(\Lambda)$.

e.g.,

because



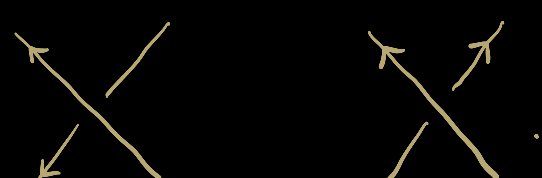
② Thurston–Bennequin number

Computed in front projection.

$$tb(\Lambda) := \operatorname{writhe}(F(\Lambda)) - \#\begin{pmatrix}\text{right cusps}\\ \text{of } F(\Lambda)\end{pmatrix}$$

$$= \#\begin{pmatrix}\text{positive crossings}\\ -\text{negative crossings}\end{pmatrix} - \#\begin{pmatrix}\text{right cusps}\\ \text{of } F(\Lambda)\end{pmatrix},$$

where



positive negative.

(Positive if moving from overstrand to understand is CCW)

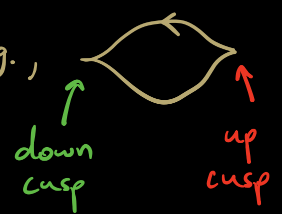
Exercise: Check that $tb(\Lambda)$ is invariant under moves given above.

Exercise: Find a formula for $tb(\Lambda)$ in terms of $\Pi(\Lambda)$.

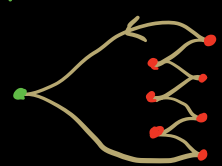
(3) Rotation number

Again, computed in $F(\Lambda)$.

$$r(\Lambda) = \frac{1}{2}\left(\# \binom{\text{down}}{\text{cusps}} - \# \binom{\text{up}}{\text{cusps}} \right)$$

e.g., 

$\longrightarrow \; r(\Lambda) = \frac{1}{2}(1-1) = 0$



$r(\Lambda) = \frac{1}{2}(1-7) = \frac{1}{2}(-6) = -3$

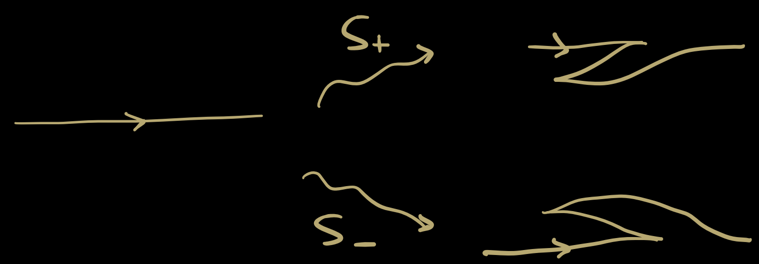
Exercise: Check that $r(\Lambda)$ is invariant under moves given above.

Exercise: Find a formula for $r(\Lambda)$ in terms of $\Pi(\Lambda)$.

Hard problem: Find Λ, Λ' which are not Leg. isotopic, but have

$$k(\Lambda) = k(\Lambda')$$
$$tb(\Lambda) = tb(\Lambda')$$
$$r(\Lambda) = r(\Lambda').$$

Stabilization. Given Λ, we can produce Legendrian knots $S_\pm(\Lambda)$, both of which are smoothly isotopic to Λ, but neither of which is Legendrian isotopic to Λ. We do this via a modification of $F(\Lambda)$:



Exercise. Compute $tb(S_\pm(\Lambda))$ and $r(S_\pm(\Lambda))$. Deduce that $S_\pm(\Lambda) \not\simeq \Lambda$.