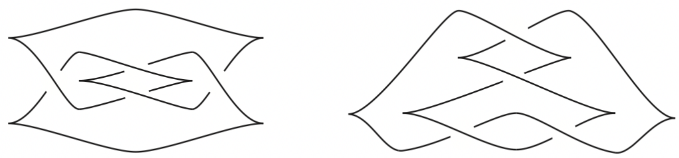
# 1 June 5 Exercises

1. Consider the parametrization $(x(t), z(t)) \in \mathbb{R}^2_{x,z}$ given by $(x(t), z(t)) = (-t^2, t^3)$, $-\infty < t < \infty$.

   (a) Find a function $y(t)$ such that $(x(t), y(t), z(t)) \in \mathbb{R}^3_{x,y,z}$ is a Legendrian parametrization. That is, your parametrization should satisfy $z'(t) - y(t)x'(t) = 0$. (But it won't be a knot.)

   (b) Sketch the front and Lagrangian projections of your Legendrian parametrization.

2. Find a Legendrian isotopy, using Legendrian Reidemeister moves, between the following front projections of the figure eight knot:

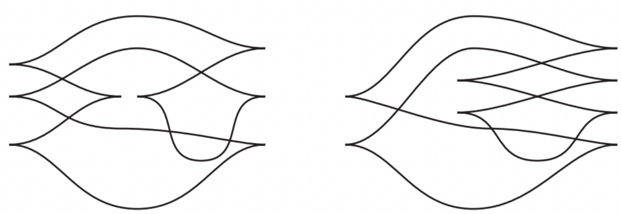


3. For each Legendrian knot $\Lambda$ whose front projection is given on this page (including below), produce the Lagrangian projection of a knot which is Legendrian isotopic to $\Lambda$. How is this a different task from producing the Lagrangian projection of $\Lambda$?
   *Hint*: For the follow-up question, it might help to revisit your sketches in Exercise 1.

4. Show that the Thurston-Bennequin invariant and rotation numbers, as defined using the front projection, are invariants of Legendrian knots.

5. Produce formulas for $\mathrm{tb}(\Lambda)$ and $\mathrm{r}(\Lambda)$ which can be computed from the Lagrangian projection $\Pi(\Lambda)$.

6. Prove that the Legendrian knots pictured below are isotopic as smooth knots, and have the same Thurston-Bennequin and rotation numbers. We will refer to these Legendrians as *the Chekanov pair*.



7. Given a Legendrian knot $\Lambda$, let $S_\pm(\Lambda)$ denote the positive (respectively, negative) stabilization of $\Lambda$. Compute $\mathrm{tb}(S_\pm(\Lambda))$ and $\mathrm{r}(S_\pm(\Lambda))$ in terms of $\mathrm{tb}(\Lambda)$ and $\mathrm{r}(\Lambda)$.

## 2 June 6 Exercises

1. Prove the product rule for total differentials: $d(fg) = g\,df + f\,dg$, for any smooth functions $f, g : \mathbb{R}^3 \to \mathbb{R}$.

2. Verify that exterior differentiation satisfies the **Leibniz rule**:

$$d(\lambda \wedge \eta) = d\lambda \wedge \eta + (-1)^{\deg(\lambda)} \lambda \wedge d\eta,$$

   for any $k$-form $\lambda$ and any $\ell$-form $\eta$. (You should only have to check a few cases.)

3. In class you were given a formula for computing the exterior derivative using a commutative diagram. (For a refresher, look at page 92 of *A Geometric Approach to Differential Forms*.) Use the commutative diagram and your knowledge of vector calculus to explain why $d \circ d : \Omega^0(U) \to \Omega^2(U)$ and $d \circ d : \Omega^1(U) \to \Omega^3(U)$ both vanish.

4. The isomorphism $\Phi_2$ (from the commutative diagram from class) shows how the curl can be interpreted as a derivative operation for vector fields on $U \subseteq \mathbb{R}^3$. Imagine that you want to compute an antiderivative for this operation, which we will call an anticurl. By definition, the anticurl of a vector field $\vec{B}$ is a vector field whose curl is $\vec{B}$. Complete the following sentence. *Any two anticurls of a vector field $\vec{B}$ can differ by at most* ______________________________________________________.

5. Complete problem **7.23** on page 94 of *A Geometric Approach to Differential Forms*. The problem is reproduced here.
   Let $C$ be any curve in $\mathbb{R}^3$ from $(0,0,0)$ to $(1,1,1)$. Let $\vec{F}$ be the vector field $\langle yz, xz, xy \rangle$. Show that $\int_C \vec{F} \cdot \vec{ds}$ does not depend on $C$.
   *Note:* The problem of finding an anti-exterior-derivative is closely related to the notion of a conservative force field. The existence of an anti-exterior-derivative is highly dependent on the shape of the domain $U$, which is measured by De Rham cohomology.

6. Fill in all the boxes in the following diagram using the diagram from class and your knowledge of vector calculus. The symbol $\times$ means either the Cartesian product of sets or the Cartesian product of functions, depending on the context.

$$\begin{tikzcd}
\Omega^1(U) \times \Omega^1(U) \arrow[r, "\wedge"] \arrow[d, "\square\times\square"'] & \Omega^{\square}(U) \arrow[d, "\square"] \\
\mathfrak{X}(U) \times \mathfrak{X}(U) \arrow[r, "\square"] & \mathfrak{X}(U)
\end{tikzcd}$$

   Hint: Start with the bottom row of the diagram. There's only one meaningful way to combine two vector fields and get another vector field in three dimensions.

7. Use the commutative diagram from the previous exercise to explain why the wedge product of any differential 1-form with itself gives zero: $\xi \wedge \xi = 0$ for any $\xi \in \Omega^1(U)$.

8. Let $\alpha = dz - y\,dx$.
   (a) Check that the differential equation $\alpha(\dot{\gamma}(t)) = 0$ is equivalent to $z'(t) - y(t)\,x'(t) = 0$.
   (b) Compute $\alpha \wedge d\alpha$.

9. (Challenge) Let $\mathcal{S} \subset \mathbb{R}^3$ be the surface parametrized by

$$F(u,v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad 0 \le u \le \pi, 0 \le v \le 2\pi.$$

   Show that

$$\int_{\mathcal{S}} \nu = 4\pi,$$

   where

$$\nu = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

   Can you identify the surface $\mathcal{S}$?

10. Let $\mathcal{B}$ be the ball of radius 1 centered at the origin in $\mathbb{R}^3$, and compute

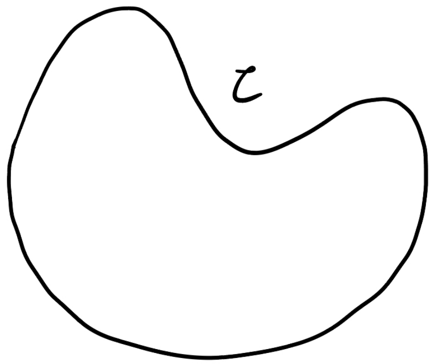
$$\int_{\mathcal{B}} d\,\nu,$$

where $\nu$ is the 2-form identified in the previous exercise.
*Hint: You shouldn't need to evaluate a triple integral if you remember the volume of a ball.*

## 3 June 7 Exercises

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := -y$, and let $f|_{S^1}$ be its restriction to the circle $S^1 = \{(x, y) \,|\, x^2 + y^2 = 1\}$. Compute the Morse DGA associated to $f|_{S^1}$.
*Note: An important part of this problem is to determine what it means to compute a DGA — i.e., how much information does your answer need to include? Also, a sketch of* $\mathrm{grad}(f|_{S^1})$ *should suffice, in leiu of an explicit calculation.*

2. Consider the curve $\mathcal{C} \subset \mathbb{R}^2$ sketched here:




Compute the Morse DGA associated to $f|_{\mathcal{C}}$, where we once again have $f(x, y) = -y$.

3. Consider the function $f_t : \mathbb{R} \to \mathbb{R}$ defined by

$$f_t(x) = x^3 + tx,$$

where $t \neq 0$ is an arbitrary real number. Compute the Morse DGA associated to $f_t$. How does the case $t < 0$ compare to the case $t > 0$? Do you see any connection to exercises 1 and 2?
*Hint: Sketching* $\mathrm{grad}(f_t)$ *(a vector field on* $\mathbb{R}$*) should help with the last question.*

4. Let $\Lambda_1, \Lambda_2$ be the Chekanov pair, whose front projections are given in the June 5 exercises. With basepoint at the bottommost point of each front projection, compute the grading of each generator of the Chekanov-Eliashberg DGA for $\Lambda_1, \Lambda_2$.

5. Prove that the Chekanov-Eliashberg DGA is not invariant under Legendrian isotopy.
*Note: You can do this before knowing how to compute the differential!*

6. We will eventually show that the Legendrian contact homology of a stabilized Legendrian knot is always trivial. Using Ng's resolution, show that there is a certain type of "gradient flow line" that we can always find when computing the Chekanov-Eliashberg DGA of a stabilized Legendrian knot.

# 4 June 8 Exercises

1. Show that any abelian group $G$ can be made into a $\mathbb{Z}$-module as follows: first, write the group operation of $G$ as $+$. Then, for any $n \in \mathbb{Z}$ and $g \in G$, define

$$ng = \begin{cases} g + g + \cdots + g \text{ ($n$ times)}, & n > 0 \\ 0_G, & n = 0 \\ -g - g - \cdots - g \text{ ($-n$ times)}, & n < 0 \end{cases},$$

where $0_G$ is the additive identity of $G$. Check the axioms for this construction to be a $\mathbb{Z}$-module.

2. Let $A$ be an associative algebra over a commutative, unital ring $R$. Define the *center* $C_A$ of $A$ as follows:

$$C_A := \{c \in A \,|\, a \cdot c = c \cdot a, \text{ for all } a \in A\}.$$

Verify that $C_A$ is an algebra over $R$.

3. Let $X$ be (the boundary of) a cube in $\mathbb{R}^3$. For concreteness, you can take $X$ to be the boundary of $[0,1] \times [0,1] \times [0,1]$, but this isn't necessary. Next, let $\mathcal{C}_0$ be the $\mathbb{Z}$-module freely generated by the vertices $v_1, v_2, \ldots, v_8$ of $X$. That is, $\mathcal{C}_0$ consists of formal linear combinations of $v_1, v_2, \ldots, v_8$, with coefficients coming from $\mathbb{Z}$. Similarly, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the free $\mathbb{Z}$-modules freely generated by the edges and faces of $X$, respectively.

   (a) For $k = 1$ and $k = 2$, define a map $\partial_k : \mathcal{C}_k \to \mathcal{C}_{k-1}$ which corresponds to taking a boundary. For instance, the boundary of a face should be a linear combination of four edges. To define $\partial_k$, you will need to choose an orientation for each edge and face. (It makes sense to orient each face counterclockwise when viewed from outside the cube.) By "define a map," we just mean that you should specify what it does to each generator of $\mathcal{C}_k$.

   (b) Produce a matrix representative for each of $\partial_1$ and $\partial_2$. Your matrix for $\partial_2$ should be $12 \times 6$, and your matrix for $\partial_1$ should be $8 \times 12$. Use these matrices to verify that $\partial_1 \circ \partial_2 = 0$.
   *Warning: Your matrix representatives will depend on your labeling of the cube; so there are many correct answers.*

   (c) The previous part shows that the image of $\partial_2$ is a subset of the kernel of $\partial_1$. (Make sure you believe this.) Show that these two $\mathbb{Z}$-submodules of $\mathcal{C}_1$ are in fact equal.
   *Hint: Identify bases for* image$(\partial_2)$ *and* ker$(\partial_1)$.

   (d) With what we've done so far, we could define $\mathcal{C}_\bullet$ to be the $\mathbb{Z}$-module freely generated by the vertices, edges, and faces of $X$, and $\partial : \mathcal{C}_\bullet \to \mathcal{C}_\bullet$ would be a linear map which decreases grading by 1 and squares to zero. (We can just define the boundary of any vertex to be zero.) Why is $\mathcal{C}_\bullet$ not a DGA?

4. Let $\mathcal{R} = \mathbb{R}^2 - \{(0,0)\}$ and consider the 1-form $\lambda \in \Omega^1(\mathcal{R})$ defined by

$$\lambda = -\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy.$$

   (a) Verify that $d\lambda = 0$.

   (b) Next, we want to show that $\lambda$ is *not* the total differential of some function. We'll do this in steps.

      i. Let $C$ be parametrized by $\mathbf{r}(t) = (\cos 2\pi t, \sin 2\pi t)$, $0 \le t \le 1$. Compute $\int_C \lambda$.

      ii. (Challenge) With the same $C$, show that if $f : \mathcal{R} \to \mathbb{R}$ is a smooth function, then

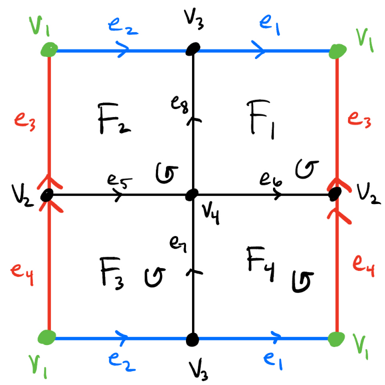
$$\int_C df = 0.$$

      *Hint: At some point you'll need the following version of the chain rule:* $\frac{d}{dt}(f(\mathbf{r}(t))) = \nabla f \cdot \mathbf{r}'(t)$.

      iii. Use the previous two parts to conclude that $\lambda \neq df$ for any function $f : \mathcal{R} \to \mathbb{R}$.

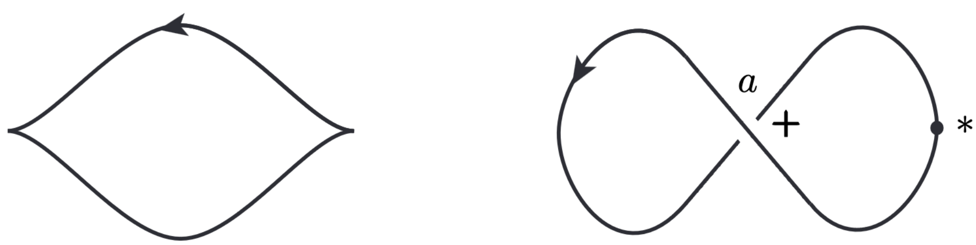
This problem demonstrates that, for the deRham differential $d$, image$(d)$ can be a strict subset of ker$(d)$; the difference between these two sets is determined by the topology of $\mathcal{R}$.

5. The following is a representation of the torus $T^2$:




We obtain a surface in $\mathbb{R}^3$ by identifying the blue horizontal edges in a way that respects their orientations, and doing the same for the red vertical edges. Just as we did for the cube in problem 3, we break $T^2$ into faces; these faces have edge boundaries, and the edges have vertex boundaries. The orientations are as labeled. (Notice the repeated labels! Those are intentional!) Repeat as much of problem 3 as you can, with $T^2$ playing the role previously played by $X$.

6. In class we computed the (linearized) Legendrian contact homology for the Legendrian unknot. The Legendrian unknot is shown in the figure below; the front projection is on the left of the figure and the Lagrangian projection of the knot is on the right.




(a) Compute the rotation number of this knot.

(b) Compute the path $\gamma_1$ for the crossing $a_1$, labeled $a$.

(c) Verify that $|t| = 0$ and $|a| = 1$ for the Chekanov-Eliashberg DGA associated to this knot.

7. In class we computed the (linearized) LCH for the Legendrian knot mirror$(5_2)$. In so doing, we wrote the differential $\partial_1 : (\mathbb{Z}_2)^4 \to (\mathbb{Z}_2)^3$ as a matrix.

$$\partial_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Find the correct sequence of row and column operations that will reduce this matrix to echelon form. At each step, keep track of what the row or column operation does to the basis in the domain and the basis of the codomain of $\partial_1$.

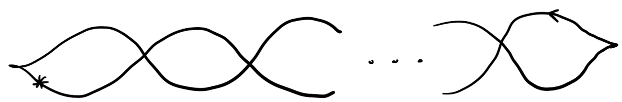
8. Use the bases that you computed in the last exercise to write an explicit module generator for $H_1(\Lambda)$ for the Legendrian mirror$(5_2)$. Recall that we found $H_1 \cong (\mathbb{Z}_2)^1$.

# 5 June 9 Exercises

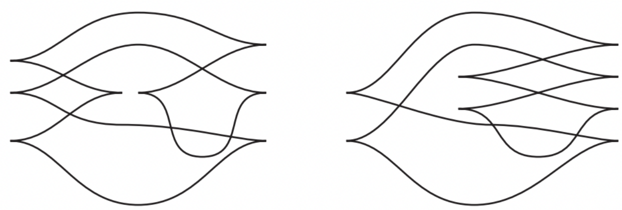
1. Devise an algorithm for determining the generators and gradings of $\mathcal{A}_\Lambda$ from the front projection $F(\Lambda)$. *Note: You can check your work by consulting Etnyre-Ng.*

2. For the Legendrian trefoil presented in class, use your algorithm to verify the gradings $|a_i|$. Then compute whichever differentials $\partial_\Lambda a_i$ we did not compute in class.

3. Compute $(\mathcal{A}_\Lambda, \partial_\Lambda)$ for the Legendrian unknot whose front projection is seen here:



4. Consider the Chekanov pair, with front projections given by



   Add a basepoint to the bottom arc of each knot, and orient the top arc left-to-right. Then compute $(\mathcal{A}_\Lambda, \partial_\Lambda)$ for each knot.

5. Suppose that $\Lambda'$ is obtained from $\Lambda$ by applying a Legendrian Reidemeister I move in the front projection. Describe how to obtain $(\mathcal{A}_{\Lambda'}, \partial_{\Lambda'})$ from $(\mathcal{A}_\Lambda, \partial_\Lambda)$.
   *Note: You may want to start with the case where the Legendrian Reidemeister move is applied along a topmost arc of $F(\Lambda)$.*

The last two exercises are intended to stimulate open-ended discussion.

6. Discuss what it will mean to filter a Legendrian knot $\Lambda$ by action. Assuming that each Reeb chord in exercise 2 has action 1, try to sketch $\Lambda_{k/2}$, $k = -1, 0, 1, 2, 3$.
   *Hint: You should start by convincing yourself that the action of a Reeb chord is given by its* height.

7. A common strategy for simplifying the computation of $(\mathcal{A}_\Lambda, \partial_\Lambda)$, given $F(\Lambda)$, is to apply Legendrian Reidemeister moves until all left cusps of $F(\Lambda)$ have the same $x$-coordinate, as do all right cusps of $F(\Lambda)$. We say that the resulting $F(\Lambda)$ is in *Legendrian plat position*. In general, this will introduce more generators for $\mathcal{A}_\Lambda$. How could this make the computation of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ simpler?

# 6 June 12 Exercises

1. In class, we saw a concrete parametrization of the Legendrian unknot.

$$\gamma(t) = (x(t), y(t), z(t)) = (\cos(t), \frac{3}{2}\cos(t)\sin(t), \frac{-1}{2}\sin^3(t)).$$

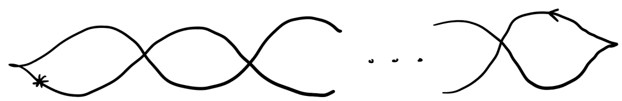
The domain is $0 \le t \le 2\pi$. You are strongly encouraged to plot this curve in three-space.

   (a) Translate each coordinate by 10 units, e.g. $(x, y, z) \mapsto (x+10, y, z)$, and similarly for $y$ and $z$. Which of these translations results in a Legendrian knot?

   (b) Project a Legendrian knot onto just a single axis, which will give a compact interval as the image. Which projection, if any, must contain the origin? Example: Project the knot parametrized by $\gamma(t)$, above, onto just the $x$-axis, and you get the interval $[-1, 1]$.

   (c) Make sure you understand how to compute the height of a Reeb chord using a definite integral.

2. Can you generate a three-dimensional parametrization of the Legendrian unknot that has $n$ crossings in its front projection? The front projection is shown here.



Hint: The case $n = 0$ should be easy.

3. Give the Lagrangian projection of a knot any orientation. Prove that the Lagrangian projection must contain signed area zero. (This means the signed area enclosed by the entire knot, not just a single lobe.)

4. Suppose you are given the image of the Lagrangian projection of a knot, and you don't necessarily have a parametrization. Devise a method for assigning an action (Reeb height) to each crossing in the Lagrangian projection. In particular:

   (a) You should understand what happens to the height assignments when the knot is dilated according to the Legendrian isotopy $(x, y, z) \mapsto (x, s \cdot y, s \cdot z)$, $1 \le s \le 2$.

   (b) Using Legendrian dilations, can you ensure that all actions that you assign are $\ge 1$ and at least one crossing gets assigned action $= 1$?

   (c) In problem 1(a) of this problem set, you discovered which translations result in Legendrian isotopies. What happens to your assignment of action if a knot undergoes a Legendrian translation?

5. Re-visit problem 6 from Friday's problem set, now that you know how to compute the action explicitly. You should now be able to generate (computer generate) plots of the filtered manifold.

6. For any Legendrian knot $\Lambda$, and for $\ell \ge 0$ define $\mathscr{A}_\Lambda^\ell$ to be the subset of $\mathscr{A}_\Lambda$ that has generators of action (Reeb length) at most $\ell$.

   (a) Check that $\mathscr{A}_\Lambda^\ell$ is a graded sub-module of $\mathscr{A}_\Lambda$.

   (b) Check that $\mathscr{A}_\Lambda^\ell$ is a unital sub-algebra of $\mathscr{A}_\Lambda$. Recall that $t$ is defined to be height 0.

# 7 June 13 Exercises

1. Let $\Lambda \subset \mathbb{R}^3$ be a Legendrian knot. We say that a Reeb chord $c$ of $\Lambda$ is *contractible* if there is a Legendrian isotopy $\Lambda_s^c$, $s \in [0,1]$, such that

   (i) $\Lambda_0^c = \Lambda$;

   (ii) the Reeb chord of $\Lambda_1^c$ which has smallest action corresponds to $c$;

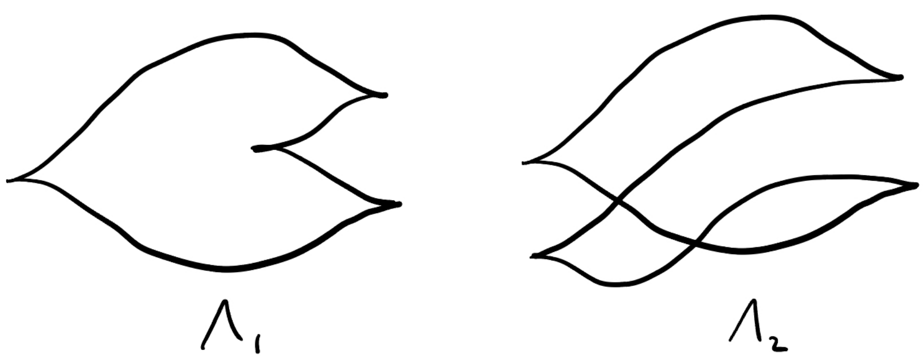
   (iii) the isotopy $\Lambda_s^c$ corresponds to a planar isotopy of $\Pi(\Lambda)$ — that is, no Reidemeister moves are allowed.

   For our usual Legendrian trefoil (c.f. Figure 6 of Etnyre-Ng), determine which of the five Reeb chords are contractible.

2. Consider the Legendrian unknot $\Lambda$ whose front projection is depicted in exercise 2 of yesterday's problem set, with $n$ crossings. Identify the $n+1$ Reeb chords of $\Lambda$ *in the front projection* — i.e., without translating to a Lagrangian projection. Describe a planar isotopy of $F(\Lambda)$ which will ensure that the Reeb chords have distinct actions.

3. Show that if $\Lambda$ is a stabilized Legendrian knot, then the Legendrian contact homology of $\Lambda$ is trivial. *Hint: Stabilization will introduce one new Reeb chord; think about this Reeb chord in the front projection to convince yourself that its action can be made arbitrarily small.*

4. Consider the Legendrian knots $\Lambda_1$ and $\Lambda_2$ whose front projections are given as follows:

   Identify a stable tame isomorphism $\Phi \colon (\mathcal{A}_{\Lambda_1}, \partial_{\Lambda_1}) \to (\mathcal{A}_{\Lambda_2}, \partial_{\Lambda_2})$.

5. Consider the DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$ computed for the Legendrian trefoil in exercise 5 of last Friday's problem set. By setting $t = 1$ and reducing all coefficients modulo 2, we obtain a DGA over $\mathbb{Z}_2$. Find all (0-graded) augmentations of this DGA to $\mathbb{Z}_2$. (There are five.)

6. Repeat the above problem for the Chekanov-Eliashberg DGAs of the Chekanov pair, as previously computed. (The two knots will not have the same number of augmentations.)

# 8 June 14 Exercises

1. Compute the following two tensor products, as interval modules.
   (a) $\mathbb{Q}[3,4)\otimes\mathbb{Q}[1,8)\cong$
   (b) $\mathbb{Q}[\sqrt{2},\infty)\otimes\mathbb{Q}[\pi,\infty)\cong$
   (c) What is a module generator of each module above?
2. Write down the persistence module of DGAs that is the Chekanov-Eliashberg DGA corresponding to the filtered Legendrian trefoil. Your filtration corresponds to the pictures that you drew for exercise 6 on June 9. Diagram this as a direct sum of interval modules.
   Note: You won't be able to compute persistent homology yet, but you can still make informative diagrams.
3. Recall that it is currently unclear how you assign heights to an un-parametrized knot diagram. On Monday (June 12) you started discussing this problem.
   (a) In class I will draw the Lagrangian projection of the Legendrian trefoil, with a specific labeling of the crossings. Check that area positivity gives the following inequalities on heights. (Abbreviate Reeb height$(a_1)=h_1$.)
   $$h_1>0$$
   $$h_2>0$$
   $$h_1-h_3-h_4-h_5>0$$
   $$-2h_1+h_2+h_3+h_5>0$$
   $$h_3+h_4>0$$
   $$h_4+h_5>0$$
   (b) How could you assign heights $h_1,\ldots,h_5$ that satisfy the above inequalities? One method is guess-and-check.
   (c) Do this again for the Legendrian Hopf Link. (This is a link, not a knot, but you still can assign heights using contact geometry!)
4. Filter the unit 2-sphere $x^2+y^2+z^2=1$ by height in the $z$-coordinate. In this exercise you will compute persistent Morse homology for this filtered manifold. Use ground ring $=\mathbb{R}$. A guide follows.
   - You should be able to draw the sub-level set of all points having height $\leq \ell$ if you are given $\ell$.
   - The entire manifold $M=S^2$ has exactly one critical point of degree 2 and exactly one critical point of degree 0. These critical points correspond to the global minimum and the global maximum of the height function. Identify these points on the filtered manifold(s), and give these points names.
   - Write down the free persistence modules that make up the Morse chain complex. Generators correspond geometrically to the points that you named.
   - Since there are no critical points of index 1, all the Morse chain complex differentials must be zero maps. Hence $d_{\cdot}^{\cdot}=0$ for every $\mathbb{Z}$-grading and every $\mathbb{R}$-grading.
   - You should now be able to compute persistent Morse homology as a direct sum of graded modules.
5. Re-compute the homology of the boundary cube from exercise 3 on June 8. This time, you will compute persistent homology by filtering the surface of the cube by height. Imagine that the surface of the cube is embedded in $\mathbb{R}^3$ in a way that is "tilted" so that the global height minimum is a single vertex. Similarly the global height maximum should be a single vertex. For the ground ring, use $\mathbb{Q}$.
   - You should be able to formulate the (filtered) chain complex as a chain complex of (direct sums of) free persistence modules.
   - The chain complex differential is an $\mathbb{R}$-filtered collection of $\mathbb{Q}$-linear transformations.
   - You should be able to formulate persistent homology as a direct sum of interval modules.

# 9 June 15 Exercises

Throughout this problem set, treat $(\mathcal{A}_\Lambda, \partial_\Lambda)$ as a DGA over $\mathbb{Z}_2$ (rather than $\mathbb{Z}$).

1. Prove that if the Chekanov-Eliashberg DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is obtained by starting with a front projection, applying Ng's algorithm to produce a Lagrangian projection, and then using the Reeb chords of this Lagrangian projection to generate $\mathcal{A}_\Lambda$, then $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is not augmented.

2. Verify that $(\mathcal{A}_\Lambda^\epsilon, \partial_\Lambda^\epsilon)$, as defined in class, is augmented.

3. Let $\Lambda$ be the Legendrian trefoil given in Figure 6 of Etnyre-Ng, and let $\epsilon$ be your favorite of the five augmentations of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ to $\mathbb{Z}_2$ identified yesterday. Compute $LCH_*^\epsilon(\Lambda)$ and $LCH_\epsilon^*(\Lambda)$.

4. Prove that the Legendrian knots which make up the Chekanov pair (c.f. June 5 exercises) are not Legendrian isotopic.
   *Note: It is not sufficient to observe that the Chekanov-Eliashberg DGAs of these knots have distinct numbers of augmentations.*

5. Let $\Lambda$ be a standard Legendrian unknot, as parametrized in the June 12 exercise set. Using the action(s) computed from that parametrization, we may filter $(\mathcal{A}_\Lambda, \partial_\Lambda)$ to produce $(\mathcal{A}_\Lambda^\ell, \partial_\Lambda^\ell)$ for any $0 \leq \ell \leq \infty$, as in class on Monday. Use this filtration, along with the augmentation $\epsilon\colon (\mathcal{A}_\Lambda, \partial_\Lambda) \to (\mathbb{Z}_2, 0)$ from class today, to compute the persistent linearized Legendrian contact homology of $\Lambda$.

6. Repeat the above exercise for the Legendrian trefoil presented in class yesterday, using some choice of heights $h_1, \ldots, h_5$ which you made in yesterday's problem session.
   *Warning: I (Austin) have not actually attempted this computation.*

# 10 June 16 Exercises

1. Compute the persistent linearized Legendrian contact homology of the standard Legendrian trefoil two more times.

   (a) Use a different assignment of heights than was done in class, and different than you used yesterday.

   (b) Try an assignment that gives each generator a different height.

   In either of the above computations, you should find that the Poincaré-Chekanov polynomial for the standard trefoil is $2+t$. The only thing that changes is the left-endpoints of the interval modules that make up your persistent homology picture. We conjecture that there is some equivalence relation that will treat all the pictures as equivalent, no matter your assignment of Reeb heights.

2. A Hopf Link can be realized as a standard Legendrian unknot that has been translated some distance away from another standard unknot. Which direction is the most natural for translation, given your answer to the isotopy exercise? With this definition, can you draw the Lagrangian projection of a Hopf link having all Reeb heights equal?

## 11 July 7 Exercises

1. You can dualize the $A_\infty$ relation to get a relationship on $\partial_j$. For each $\ell \geq 1$, we have

$$0 = \sum_{i+j+k=\ell} (\mathbb{1}^{\otimes i} \otimes \partial^{\epsilon}_{(j)} \otimes \mathbb{1}^{\otimes k}) \circ \partial^{\epsilon}_{(i+1+k}.$$

   Warning: This sum is not taken over all possible combinations of $i, j, k$. Instead, you can only add operators that have the same domain and the same codomain.

   Check that this relationship holds for $\ell = 1, 2, 3$ for the $\partial^{\epsilon}_{(\ell)}$ defined for the right-handed trefoil. Remember that we use $\mathbb{Z}_2$ coefficients!

2. Find a height assignment for the knot that Austin described in his example from class. Draw the barcode for the knot, and consider what happens to heights when you multiply various Reeb chords.