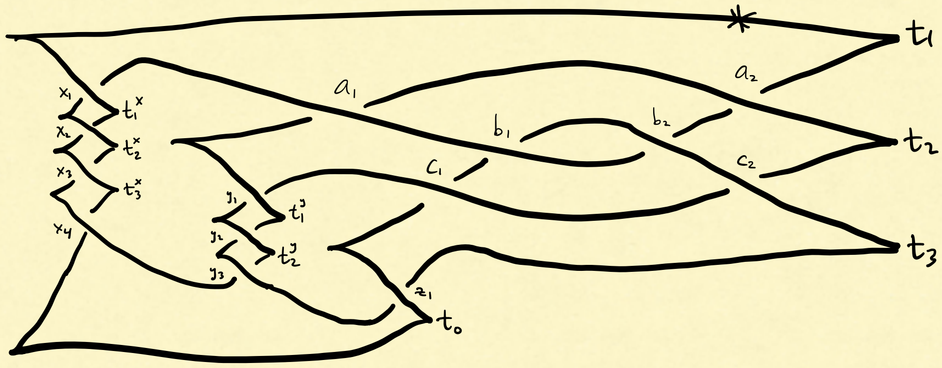
# Legendrian knots distinguished via $m_2$

Ref: "Product structures for Legendrian contact homology", Civan, et al

Consider $\Lambda$ with the following front projection:




In $(\mathcal{A},\partial)$, gradings are:

$$|t_i| = |t_i^x| = |t_i^y| = |t_i^z| = 1$$
$$|a_1| = |a_2| = 0$$
$$|b_1| = -|b_2| = 2$$
$$|c_1| = -|c_2| = 1$$
$$|x_i| = |y_i| = |z_i| = 0.$$

The differential is given by

$\partial a_1 = 0$ $\qquad \partial b_1 = a_1 y_1 c_1$ $\qquad \partial c_1 = 0$

$\partial a_2 = y_1 c_1 b_2$ $\qquad \partial b_2 = 0$ $\qquad \partial c_2 = b_2 a_1 y_1$

$$\partial t_1 = 1 + x_1(1 + a_1 a_2 + b_1 b_2)$$
$$\partial t_2 = 1 + (1 + a_2 a_1) y_1 (1 + c_1 c_2)$$
$$\partial t_3 = 1 + (1 + b_2 b_1 + c_2 c_1) z_1$$
$$\partial t_0 = 1 + x_4 y_3 z_1$$

$$\partial t_i^x = 1 + x_i x_{i+1}, \quad 1 \le i \le 3$$
$$\partial t_i^y = 1 + y_i y_{i+1}, \quad 1 \le i \le 2$$
$$\partial x_i = \partial y_i = \partial z_i = 0.$$

$(\mathcal{A},\partial)$ admits a unique augmentation $\varepsilon : \mathcal{A} \to \mathbb{Z}_2$

$\varepsilon(x_i) = \varepsilon(y_i) = \varepsilon(z_i) = 1$, all other generators map to $0$

$$\Phi^{\varepsilon}(a) = a + \varepsilon(a) \quad ; \quad \partial^{\varepsilon} = \Phi^{\varepsilon} \circ \partial \circ (\Phi^{\varepsilon})^{-1}$$

$$\partial^{\varepsilon} a_1 = 0$$
$$\partial^{\varepsilon} a_2 = c_1 b_2 + y_1 c_1 b_2$$
$$\partial^{\varepsilon} b_1 = a_1 c_1 + a_1 y_1 c_1$$
$$\partial^{\varepsilon} b_2 = 0$$
$$\partial^{\varepsilon} c_1 = 0$$
$$\partial^{\varepsilon} c_2 = b_2 a_1 + b_2 a_1 y_1$$
$$\partial^{\varepsilon} t_1 = x_1 + a_1 a_2 + b_1 b_2 + x_1 a_1 a_2 + x_1 b_1 b_2$$
$$\partial^{\varepsilon} t_2 = y_1 + a_2 a_1 + c_1 c_2 + a_2 a_1 y_1 + y_1 c_1 c_2 + a_2 a_1 c_1 c_2 + a_2 a_1 y_1 c_1 c_2$$
$$\partial^{\varepsilon} t_3 = z_1 + b_2 b_1 + c_2 c_1 + b_2 b_1 z_1 + c_2 c_1 z_1$$
$$\partial^{\varepsilon} t_0 = x_4 + y_3 + z_1 + x_4 y_3 + y_3 z_1 + x_4 z_1 + x_4 y_3 z_1$$
$$\partial^{\varepsilon} t_i^x = x_i + x_{i+1} + x_i x_{i+1}$$
$$\partial^{\varepsilon} t_i^y = y_i + y_{i+1} + y_i y_{i+1}$$

$\dim A = 23$. rank $\partial^{\varepsilon}_{(1)} = 8 \Rightarrow \operatorname{Ker} \partial^{\varepsilon}_{(1)}$ has dim. 15

$\operatorname{Ker} \partial^{\varepsilon}_{(1)}$ is spanned by $\overbrace{a_i, b_i, c_i, x_i, y_i, z_i}^{14}$, and

$t := \Sigma_i t_i + \Sigma_i t_i^x + \Sigma_i t_i^y + \Sigma_i t_i^z$.

$(\partial^{\varepsilon}_{(1)})^2 = 0$ ; rank $\partial^{\varepsilon}_{(1)} = 8 \Rightarrow \operatorname{Ker} \partial^{\varepsilon}_{(1)} / \operatorname{im} \partial^{\varepsilon}_{(1)} \cong \mathbb{Z}_2^7$

In fact, $\operatorname{im} \partial^{\varepsilon}_{(1)} = \mathbb{Z}_2 \langle x_1, x_2, x_3, x_4, y_1, y_2, y_3, z_1 \rangle$,

so $\dfrac{\operatorname{Ker} \partial^{\varepsilon}_{(1)}}{\operatorname{im} \partial^{\varepsilon}_{(1)}} = \mathbb{Z}_2 \langle [a_i], [b_i], [c_i], [t] \rangle$.

Now $\partial^{\varepsilon}_{(2)} : A \to A^{\otimes 2}$ is represented by a $529 \times 23$ matrix,
and $m_2 : A^{\otimes 2} \to A$ is represented by its transpose.

However, $m_2$ is very easy to read from $\partial^\varepsilon$ above. For instance, $m_2(c_1 \otimes b_2) = a_2$, because $\partial^\varepsilon a_2$ is the only term in which $c_1 b_2$ appears. Note that $m_2(x \otimes y)$ is the sum of the terms $g_1, \ldots, g_k$ for which $xy$ appears in $\partial^\varepsilon g_i$.

At the chain level, $m_2$ is given by

$$m_2(a_1 \otimes a_2) = m_2(b_1 \otimes b_2) = t_1 \qquad m_2(c_1 \otimes b_2) = a_2$$
$$m_2(a_2 \otimes a_1) = m_2(c_1 \otimes c_2) = t_2 \qquad m_2(a_1 \otimes c_1) = b_1$$
$$m_2(b_2 \otimes b_1) = m_2(c_2 \otimes c_1) = t_3 \qquad m_2(b_2 \otimes a_1) = c_2$$

$$m_2(x_i \otimes x_{i+1}) = t_i^x \qquad m_2(y_i \otimes y_{i+1}) = t_i^y$$

In homology these become

$$\langle [a_1], [a_2] \rangle = \langle [a_2], [a_1] \rangle = [t] \qquad \langle [c_1], [b_2] \rangle = [a_2]$$
$$\langle [b_1], [b_2] \rangle = \langle [b_2], [b_1] \rangle = [t] \qquad \langle [a_1], [c_1] \rangle = [b_1]$$
$$\langle [c_1], [c_2] \rangle = \langle [c_2], [c_1] \rangle = [t] \qquad \langle [b_2], [a_1] \rangle = [c_2]$$

Check by computing $\partial^\varepsilon_{(2)} t$.

To distinguish $\Lambda$ from its mirror, let's pay attention to gradings:

$$LCH^\varepsilon_{-2} = \mathbb{Z}_2 \langle [b_2] \rangle$$
$$LCH^\varepsilon_{-1} = \mathbb{Z}_2 \langle [c_2] \rangle$$
$$LCH^\varepsilon_{0} = \mathbb{Z}_2 \langle [a_1], [a_2] \rangle$$
$$LCH^\varepsilon_{1} = \mathbb{Z}_2 \langle [c_1], [t] \rangle$$
$$LCH^\varepsilon_{2} = \mathbb{Z}_2 \langle [b_1] \rangle.$$

(Notice that $|\langle [x], [y] \rangle| = |[x]| + |[y]| + 1$.)

So the right column gives products

$$LCH^\varepsilon_1 \otimes LCH^\varepsilon_{-2} \longrightarrow LCH^\varepsilon_0$$
$$LCH^\varepsilon_0 \otimes LCH^\varepsilon_1 \longrightarrow LCH^\varepsilon_2$$
$$LCH^\varepsilon_{-2} \otimes LCH^\varepsilon_0 \longrightarrow LCH^\varepsilon_{-1}.$$

The C.E. DGA $(\mathcal{A}_{\bar\Lambda}, \partial)$ for the mirror $\bar\Lambda$ has the same generators as $(\mathcal{A}_\Lambda, \partial)$, but their order is reversed in $\partial$. For instance,

$$\partial t_1 = 1 + x_1(1 + a_1 a_2 + b_1 b_2) \text{ in } \mathcal{A}_\Lambda$$
$$\Downarrow$$
$$\partial t_1 = 1 + (1 + a_2 a_1 + b_2 b_1) x_1 \text{ in } \mathcal{A}_{\bar\Lambda}.$$

As a result, the linearized LCH of $\bar\Lambda$ agrees with that of $\Lambda$, but the product on the chain level is now given by

$$m_2(a_2 \otimes a_1) = m_2(b_2 \otimes b_1) = t_1 \qquad m_2(b_2 \otimes c_1) = a_2$$
$$m_2(a_1 \otimes a_2) = m_2(c_2 \otimes c_1) = t_2 \qquad m_2(c_1 \otimes a_1) = b_1$$
$$m_2(b_1 \otimes b_2) = m_2(c_1 \otimes c_2) = t_3 \qquad m_2(a_1 \otimes b_2) = c_2$$
$$m_2(x_{i+1} \otimes x_i) = t_i^x \qquad m_2(y_{i+1} \otimes y_i) = t_i^y.$$

In homology these become

$$\langle [a_1], [a_2] \rangle = \langle [a_2], [a_1] \rangle = [t] \qquad \langle [b_2], [c_1] \rangle = [a_2]$$
$$\langle [b_1], [b_2] \rangle = \langle [b_2], [b_1] \rangle = [t] \qquad \langle [c_1], [a_1] \rangle = [b_1]$$
$$\langle [c_1], [c_2] \rangle = \langle [c_2], [c_1] \rangle = [t] \qquad \langle [a_1], [b_2] \rangle = [c_2].$$

So we now have

$$LCH^\varepsilon_{-2} \otimes LCH^\varepsilon_1 \longrightarrow LCH^\varepsilon_0$$
$$LCH^\varepsilon_1 \otimes LCH^\varepsilon_0 \longrightarrow LCH^\varepsilon_2$$
$$LCH^\varepsilon_0 \otimes LCH^\varepsilon_{-2} \longrightarrow LCH^\varepsilon_{-1}.$$

<u>Upshot.</u> $\Lambda$ and $\bar{\Lambda}$ are not Legendrian isotopic.

**Q** How is $h(m_2(x \otimes y))$ related to $h(x)$ and $h(y)$?

Can the tensor product on interval modules recover $m_2$, or at least the product structure on homology?

(Recall: $R[a,b) \otimes R[c,d) \cong R[a+c, \min\{a+d, b+c\})$.)

Suggestion: Try to run the flooding algorithm with still more conditions. Namely, try to enforce

$$h(m_2(x \otimes y)) = h(x) + h(y).$$

In the example given here, this would provide 13 more equations, which may be too many.