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Differential Graded Algebras I

Goal: In broad strokes, an algebraic topologist wants to understand the algebraic structures that are hidden inside of various topological manifolds (or spaces). Over the next two weeks, we will try to understand these two maps of sets:

$$\{\text{Legendrian knots}\} \longrightarrow \{\text{Chekanov-Eliashberg algebra}\} \longrightarrow \left\{\begin{matrix}\text{Legendrian}\\ \text{contact homology}\end{matrix}\right\}$$

The next goal will be to understand whether these sets have algebraic structure that is seen by the set maps.

Note: The middle set in the above diagram has a large amount of algebraic structure. It's an example of a <u>differential graded algebra</u> (abbr. DGA). We'll start by understanding a simpler example of a DGA, which is the DGA of differential forms on an open subset $\mathcal{U} \subset \mathbb{R}^3$.

Analogy: This DGA of differential forms fits into a diagram like the one above:

$$\{\text{open subsets } \mathcal{U} \subseteq \mathbb{R}^3\} \longrightarrow \{\text{differential forms}\} \longrightarrow \{\text{deRham cohomology of } \mathcal{U}\}.$$

For algebraic topologists, this diagram is a rich source of ideas. We'll only scratch the surface before moving on to the diagram at the top of the page.

Example: The differential one-form $\alpha = dz - y\,dx$ on $\mathbb{R}^3$ restricts to a differential one-form on any open subset $\mathcal{U} \subseteq \mathbb{R}^3$. We assert that $\alpha = dz - y\,dx$ is a diff. one-form on $\mathcal{U}$ by writing $\alpha \in \Omega^1(\mathcal{U})$.

Notation: $C^\infty(\mathcal{U})$ is the ring of all smooth real-valued functions $\mathcal{U} \to \mathbb{R}$.

First Structure: $\Omega^1(\mathcal{U})$ is a module over the ring $C^\infty(\mathcal{U})$.

- differential one-forms can be added to give another differential one form e.g. $dx \in \Omega^1(\mathcal{U})$ and $dy \in \Omega^1(\mathcal{U}) \Rightarrow dx + dy \in \Omega^1(\mathcal{U})$.
- a diff. one form can be multiplied by a smooth function to give a diff. 1-form. e.g. $dz \in \Omega^1(\mathcal{U})$ and $f(x,y,z) = x \cdot y \cdot z \in C^\infty(\mathcal{U}) \Rightarrow f \cdot dz = xyz\,dz \in \Omega^1(\mathcal{U})$.
- addition of forms has an obvious inverse operation.
- multiplication (function)$\cdot$(1-form) distributes over addition.
- the unit in $C^\infty(\mathcal{U})$ is the constant function 1, and $1 \cdot \beta = \beta \;\; \forall \beta \in \Omega^1(\mathcal{U})$.

Second structure: differential forms are graded by degree.

Recall: The module of differential forms of degree k has rank $= \binom{n}{k} = \binom{3}{k}$. A basis will be given shortly.

- differential zero-forms have rank $\binom{3}{0} = 1$. As a set, $\Omega^0(\mathcal{U}) = C^\infty(\mathcal{U})$.
- ⋮
- differential three-forms have rank $\binom{3}{3} = 1$
- Combine all these modules into a single graded module over $C^\infty(\mathcal{U})$ given by $\Omega^\bullet(\mathcal{U}) = \Omega^0(\mathcal{U}) \oplus \Omega^1(\mathcal{U}) \oplus \Omega^2(\mathcal{U}) \oplus \Omega^3(\mathcal{U})$.
- Every element of $\Omega^\bullet(\mathcal{U})$ can be written as a sum of homogeneous terms, having a single degree.

Definition: A sub-module N of $\Omega^\bullet(\mathcal{U})$ is called a graded sub-module if N has a grading that satisfies $N_j = N \cap \Omega^j(\mathcal{U})$.

Third structure: wedge product makes $\Omega^\bullet(\mathcal{U})$ into a graded algebra.

- Wedge product is a kind of multiplication $\Omega^j(\mathcal{U}) \times \Omega^k(\mathcal{U}) \xrightarrow{\wedge} \Omega^{j+k}(\mathcal{U})$ that makes $\Omega^\bullet(\mathcal{U})$ into an algebra over the ring $C^\infty(\mathcal{U})$.
- The wedge product is associative and bilinear: $dz \wedge (f\,dx + g\,dy) =$
  $= f\,dz \wedge dx + g\,dz \wedge dy$.
- It is also true that $h(\alpha \wedge \beta) = (h\alpha) \wedge \beta = \alpha \wedge (h\beta)$ for any $\alpha \in \Omega^j(\mathcal{U})$, $\beta \in \Omega^k(\mathcal{U})$ and any $h \in C^\infty(\mathcal{U})$.
- Wedge product respects the grading in the sense that $\Omega^j \wedge \Omega^k \to \Omega^{j+k}$.
- This multiplication is graded commutative, meaning that $\boxed{\alpha \wedge \beta = (-1)^{jk} \beta \wedge \alpha.}$

Example: For one-forms $dx \in \Omega^1(\mathcal{U})$ and $dz \in \Omega^1(\mathcal{U})$ the graded commutativity rule says $dx \wedge dz = -dz \wedge dx$.

Question to ponder: Do you understand how the algebra structure interacts with the previous two structures?

Fourth structure: $\Omega^\bullet(\mathcal{U})$ is a (co)chain complex

Recall: There is a map $d: \Omega^k(\mathcal{U}) \to \Omega^{k+1}(\mathcal{U})$ called the exterior derivative that satisfies $d^2 = d \circ d = 0 : \Omega^k(\mathcal{U}) \to \Omega^{k+2}(\mathcal{U})$.

This map is also called the deRham differential or the coboundary operator.

Notation: For $\mathcal{U} \subset \mathbb{R}^3$, let $\mathfrak{X}(\mathcal{U})$ denote the set of all smooth vector fields defined on $\mathcal{U}$.

Note: For $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^3$ (and for only 3D-space), it's possible to compute the exterior derivative of forms using the following commutative diagram.

$$\begin{tikzcd}
(f\in)\, C^\infty(\mathcal{U}) \arrow[r, "\text{grad}"] \arrow[d, "\text{identity}"] & \mathfrak{X}(\mathcal{U}) \arrow[r, "\text{curl}"] \arrow[d, "\Phi_1", "\cong"'] & \mathfrak{X}(\mathcal{U}) \arrow[r, "\text{div}"] \arrow[d, "\Phi_2", "\cong"'] & C^\infty(\mathcal{U}) \arrow[d, "\Phi_3", "\cong"'] \\
\Omega^0(\mathcal{U}) \arrow[r, "d"] & \Omega^1(\mathcal{U}) \arrow[r, "d"] & \Omega^2(\mathcal{U}) \arrow[r, "d"] & \Omega^3(\mathcal{U}).
\end{tikzcd}$$

Write a vector field $\vec{E} \in \mathfrak{X}(\mathcal{U})$ as a triple (E_1, E_2, E_3) of smooth functions on $\mathcal{U}$. Then define isomorphisms

$\Phi_1(E_1, E_2, E_3) = E_1\,dx + E_2\,dy + E_3\,dz$

$\Phi_2(E_1, E_2, E_3) = E_1\,dy \wedge dz + E_2\,dz \wedge dx + E_3\,dx \wedge dy$, and

$\Phi_3(h) = h\,dx \wedge dy \wedge dz$ for $h \in C^\infty(\mathcal{U})$.

The inverse maps should be obvious.

Example: (of commutativity of the first square) For $f \in C^\infty(\mathcal{U})$,
$\text{grad}(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ and $\Phi_1(\text{grad}(f)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = d(id(f))$.

Note: A (co)chain complex is the algebraic structure needed to compute (co)homology. If you do this for the example $\Omega^\bullet(\mathcal{U})$, you get $H^\bullet_{deRham}(\mathcal{U})$.

Motivating Question: What is the algebraic structure of $H^\bullet_{dR}(\mathcal{U})$?

This question could be viewed in two ways:

- What is the least amount of algebraic structure needed to start doing computations?
- What more can be added to discern interesting topological properties of $\mathcal{U}$?

The same can be asked about LCH of knots.

Note: Before taking any homology, we want to think about how the fourth structure interacts with the previous three.

- The exterior derivative is additive: $d(\alpha + \beta) = d(\alpha) + d(\beta)$.
- For $f \in C^\infty(\mathcal{U})$, $d(f \wedge dx) = \frac{\partial f}{\partial x}dx \wedge dx + \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial f}{\partial z}dz \wedge dx$.

Leibniz rule

- For any (homogeneous) differential forms $\lambda \in \Omega^j$ and $\eta \in \Omega^k(\mathcal{U})$, $\boxed{d(\lambda \wedge \eta) = d\lambda \wedge \eta + (-1)^j \lambda \wedge d\eta.}$
- The map d is said to have degree 1 because $d : \Omega^k(\mathcal{U}) \to \Omega^{k+1}(\mathcal{U})$. Gather together all of the cochain maps into a single map $d : \Omega^\bullet(\mathcal{U}) \to \Omega^\bullet(\mathcal{U})$. In this context, d is called a <u>graded module morphism of degree 1</u>.

Proposition: The kernel and the image of d are each graded submodules of $\Omega^\bullet(\mathcal{U})$.

Example: We can use all the work we've done so far to conclude that $\Omega^\bullet(\mathcal{U})$ is an example of a differential graded algebra over the ring $C^\infty(\mathcal{U})$.

Definition: A <u>differential graded algebra</u> over a commutative unital ring is a graded module over R, with a compatible algebra structure and a (co)chain complex structure that satisfies the Leibniz rule.

Hint: How do you remember the sign convention for the two rules that are boxed in red boxes? Any time you move an object of degree j past an object of degree k, it costs $j \cdot k$ negative ones. This works for d, an object of degree 1.

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DGA's II: Homology

Recall: A differential graded algebra over a commutative unital ring R is a graded module over R with a compatible algebra structure and a (co)chain complex structure that satisfies the Leibniz rule. $\Omega^{\bullet}(U)$ and $\mathcal{A}_\Lambda$ are examples of DGAs.

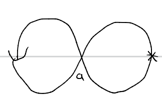
Note: Today we are going to compute homology of a DGA, but initially we won't need the algebra structure. You can compute the homology of any (co)chain complex.

Definition: Given a chain complex (possibly infinite in length) of modules,

$$\cdots \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 \xleftarrow{d_4} \cdots,$$

note that $\ker(d_i) \overset{\text{sub-mod}}{\subseteq} C_i$ and $\text{image}(d_{j+1}) \overset{\text{s-mod}}{\subseteq} C_j$. The k^{th} homology is the quotient module $H_k = \ker(d_k)/\text{image}(d_{k+1})$.

Remark: If you start with a cochain complex, and you compute the relevant kernel/image, then you are computing cohomology. For LCH we use homology.

Example: Now let's compute LCH assuming that we are given the DGA. So take for granted that the Chekanov-Eliashberg DGA associated with the Legendrian unknot  has as a chain complex $\mathbb{Z}\langle t, t^{-1}\rangle \xleftarrow{\partial} \mathbb{Z}\langle a \rangle$

degree $\quad 0 \quad\quad 1$

and all the modules in the other degrees vanish.

(Note that $r(F(\Lambda)) = \frac{1}{2}(1-1) = 0 \Rightarrow |t| = 0$ and for $a_1 = a$ the only crossing, $\text{rot}(\gamma_1) = \frac{3}{4} \Rightarrow |a| = 2\left(\frac{3}{4}\right) - \frac{1}{2} = 1.$)

The (chain complex) differential map is $\partial(a) = \cancel{1} + t^{-1} = t^{-1}$

Now we can compute (linearized) LCH from the definition above.

$H_1(\Lambda) = \text{kernel}(\mathbb{Z}\langle a \rangle \xrightarrow{\partial} \mathbb{Z}\langle t, t^{-1}\rangle)/\text{image}(0 \xrightarrow{\partial} \mathbb{Z}\langle a \rangle) \cong 0/0 = \text{rank } 0.$

$H_0(\Lambda) = \text{kernel}(\mathbb{Z}\langle t, t^{-1}\rangle \xrightarrow{\partial} 0)/\text{image}(\partial) = \langle t, t^{-1}\rangle / \langle t^{-1}\rangle \cong \mathbb{Z}\langle t \rangle = \text{rank } 0$ after linearizing.

All other modules H_2, H_3, etc. are trivial.

Note: Since we are working with modules, all of these ker/image computations can be done using matrices. The paper "Persistent Homology - a survey" explains this quite well. Let's do another example using matrices.

Example: We will compute the (linearized) LCH for the Legendrian knot mirror(5_2).

Again we will take the DGA for granted. Since this knot is more complicated, we'll use $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ as the ring of coefficients. This makes every module into a $\mathbb{Z}_2$-vector space. The DGA chain complex is

$$\overset{}{0} \xleftarrow{0} \mathbb{Z}_2\langle a_6\rangle \xleftarrow{\partial_{-1}} 0 \xleftarrow{\partial_0} \mathbb{Z}_2\langle \cancel{t, t^{-1}} a_7, a_8, a_9\rangle \xleftarrow{\partial_1} \mathbb{Z}_2\langle a_1, a_2, a_3, a_4\rangle \xleftarrow{\partial_2} \mathbb{Z}_2\langle a_5\rangle \leftarrow 0$$

Degree: $\qquad -2 \qquad\qquad -1 \qquad\qquad 0 \qquad\qquad 1 \qquad\qquad 2$

The chain complex differentials, after doing "linearization" are given by.

$\partial_1(a_1) = a_7 + \cancel{a_6}a_5 + a_7\cancel{a_6}a_5$

$\partial_1(a_2) = a_9 + \cancel{a_5}a_6 + a_5\cancel{a_6}a_9$ $\qquad$ and $\partial_\bullet(a_i) = 0$ for all $i > 4$

$\partial_1(a_3) = a_8 + a_7 + \cancel{a_8 a_7}$ $\qquad\qquad$ $\bullet$ = any degree.

$\partial_1(a_4) = a_9 + a_8 + \cancel{a_9 a_8}$

Now we can write the map ∂_1 as a matrix $\partial_1 : (\mathbb{Z}_2)^4 \to (\mathbb{Z}_2)^3$ with <u>ordered</u> bases given above

$$\begin{array}{c|c} & \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \\ \begin{matrix} a_7 \\ a_8 \\ a_9 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \xmapsto{\text{r.r.e.f.}} \begin{bmatrix} 1 & & & \pm 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{bmatrix}$$

Since the first part of the r.r.e.f matrix is the 3×3 identity, we can take a basis for $\text{image}(\partial_1)$ to be $\{a_7, a_9, a_7 + a_9\}$ using the original ordering.

But every vector space is a free module! So really we only have to keep track of $rk(\partial_1) = 3$ and $\text{nullity}(\partial_1) = 1$. Recall all other $\partial_{k\neq 1} = 0$. So we have

$H_0 = \ker(\partial_0)/\text{image}(\partial_1) \cong \mathbb{Z}_2^3 / \mathbb{Z}_2^3 \cong (\mathbb{Z}_2)^{\text{nullity}(\partial_0) - rk(\partial_1)} \cong 0.$

$H_1 = \ker(\partial_1)/\text{image}(\partial_2) \cong (\mathbb{Z}_2)^{\text{nullity}(\partial_1) - rk(\partial_2)} \cong (\mathbb{Z}_2)^1.$

$H_{-1} = 0$ because $\ker(\partial_{-1}) \subseteq 0.$

$H_2 = \ker(\partial_2)/\text{image}(0) \cong \langle a_5\rangle / 0 \cong \mathbb{Z}_2\langle a_5\rangle \cong (\mathbb{Z}_2)^1.$

$H_{-2} = \ker(\partial_{-2})/\text{image}(\partial_{-1}) \cong \langle a_6\rangle / 0 \cong \mathbb{Z}_2\langle a_6\rangle \cong (\mathbb{Z}_2)^1.$

A useful shorthand for all this information is the <u>Poincaré-Chekanov polynomial</u>

$$P(\lambda) = \sum_i \dim(H_i)\cdot\lambda^i = 1\lambda^{-2} + 1\lambda^1 + 1\lambda^2.$$

Remark: We did a lot of algebra today involving the (A_Λ, ∂) boundary maps. We still need to explain the geometry behind this boundary map.

Preview: After you understand how to compute LCH, we will add an extra complication by doing persistent homology. What does the adjective persistent mean? We want this algebraic computation (LCH) to somehow capture which topological features "persist" as we slowly build up the manifold Λ. To do this, we need to "chop up" Λ into a filtered manifold, one Λ_c for every $c \in \mathbb{R}$ satisfying

$$\dots \subseteq \Lambda_{-2} \subseteq \Lambda_{-1} \subseteq \Lambda_0 \subseteq \Lambda_{0.1} \subseteq \Lambda_{0.35} \subseteq \Lambda_{0.5} \subseteq \Lambda_{1.1} \subseteq \Lambda_\pi \subseteq \dots$$

Then we will need to improve the diagram from last time.

$$\left\{\begin{matrix}\text{Filtered} \\ \text{Legendrian} \\ \text{knots}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Filtered} \\ \text{Chekanov-} \\ \text{Eliashberg}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Persistent} \\ \text{LCH}\end{matrix}\right\}$$

One could ask: algebra multiplication $\longmapsto$???

The algebraic structure that we need for the middle set is (probably) a persistence module together with C.E. DGA structure. I'll explain persistence modules next week.