Properties of the Chekanov–Eliashberg DGA

The C.E. DGA is a DGA

Theorem (Chekanov) Let $\Lambda \subset \mathbb{R}^3$ be an oriented, pointed Legendrian knot. Then $\partial_\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ has degree -1 and satisfies $\partial_\Lambda \circ \partial_\Lambda = 0$, making $(\mathcal{A}_\Lambda, \partial_\Lambda)$ a $\mathbb{Z}$-graded DGA.

(Proof idea.) These two properties of ∂_Λ are precisely why we gave such a complicated def'n of ∂_Λ. We immediately have $\deg(\partial_\Lambda) = -1$, since

$$\Delta(a; b_1, \dots, b_n) \neq \emptyset \Rightarrow |a| = \sum_{i=1}^{n} |b_i| + 1.$$

The coefficient of $(\partial_\Lambda \circ \partial_\Lambda) a$ on some (module) generator $b_1 b_2 \cdots b_n$ counts a "broken" version of the moduli spaces we've considered. The proof of $\partial_\Lambda^2 = 0$ requires seeing this as counting the components of the bdry of a 1D moduli space. ∎

Thus the second step of

$$\left\{ \begin{array}{c} \text{oriented, pointed} \\ \text{Leg. knots} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{C.E.} \\ \text{DGAs} \end{array} \right\} \longrightarrow \{ \text{LCH} \}$$

is possible. What about equivalence relations?

The C.E. DGA is an invariant

Recall that our def'n of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ may require applying a Legendrian isotopy to obtain a nice Lagrangian projection. But Legendrian isotopy can certainly change $(\mathcal{A}_\Lambda, \partial_\Lambda)$. (Why?)

In this sense, $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is not well-defined for an arbitrary (oriented, pointed) Legendrian knot $\Lambda \subset \mathbb{R}^3$.

<u>Q</u>: What is the correct notion of equivalence for C.E. DGAs?

<u>A</u>: Stable tame isomorphisms

<u>Theorem</u> (Chekanov) Let $\Lambda \subset \mathbb{R}^3$ be an oriented Legendrian knot. The stable tame isomorphism type of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is invariant under Legendrian isotopy and choice of base point.

<u>Def</u>. An elementary automorphism of $(\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle, \partial)$ is a chain map $\phi : \mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle \to \mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle$ with the form

i.e., $\phi \circ \partial = \partial \circ \phi$

$$\phi(a_j) = \pm t^k a_j t^\ell + u, \quad \text{for some } u \in \mathbb{Z}\langle a_1, \dots, \hat{a}_j, \dots, a_n, t^{\pm 1}\rangle,\ k, \ell \in \mathbb{Z};$$

$$\phi(a_i) = a_i, \quad \text{for } i \neq j;$$

$$\phi(t) = t,$$

for some $1 \leq j \leq n$. A tame isomorphism

$$(\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle, \partial) \longrightarrow (\mathbb{Z}\langle b_1, \dots, b_n, s^{\pm 1}\rangle, \partial)$$

is obtained by composing some number of elementary automorphisms, and then applying the isomorphism

$$a_i \mapsto b_i, \quad t \mapsto s, \quad t^{-1} \mapsto s^{-1}.$$

Note that a tame isomorphism is in particular an isomorphism of DGAs, and thus cannot be what we're looking for.

Def The grading k stabilization of the DGA $(\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle, \partial)$ is the DGA with

algebra: $\mathbb{Z}\langle e_k, e_{k-1}, a_1, \dots, a_n, t^{\pm 1}\rangle$

grading: $|e_k| = k$, $|e_{k-1}| = k-1$, others unchanged

differential: $\partial e_k = e_{k-1}$, $\partial e_{k-1} = 0$, others unchanged.

Exercise: The grading k stabilization of the DGA $(\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle, \partial)$ has the same homology as $(\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle, \partial)$.

Notice that the grading k stabilization adds two generators, as we might see when applying a double point move to $\pi(\Lambda)$, while a tame isomorphism shuffles the existing generators, as does a triple point move.

Def. A pair of DGAs is stable tame isomorphic if stabilizing each of them some number of times yields DGAs which are tame isomorphic.

$$\{\text{tame isomorphisms}\} \subsetneq \{\text{isomorphisms}\}$$
$$\not\cap$$
$$\{\text{stable tame isomorphisms}^*\} \boxed{\begin{aligned}&\subseteq \{\text{chain h'topy equivalences}\}\\ &\subsetneq \{\text{quasi-isomorphisms}\}\end{aligned}}$$

Notions of equivalence which preserve homology →

In general, looking for / constructing stable tame isomorphisms is a very messy process. You'll do one example in the exercises, but we'll mostly focus on extracting simpler algebraic data from $(\mathcal{A}_\Lambda, \partial_\Lambda)$.

Some properties of the C.E. DGA

<u>Prop</u> Stabilized Legendrian knots have trivial LCH.

<u>Proof</u> is an exercise using the fact that the discs counted by ∂_Λ must decrease action.

<u>Cor.</u> Quasi-isomorphic C.E. DGAs need not be stable tame isomorphic.

(Proof.) Define $\chi((\mathbb{Z}\langle a_1, \dots, a_n \rangle, \partial))$ to be the # of generators of even degree $-$ # of generators of odd degree.

Fact: Stabilization does not change χ.

Fact: $a_i \in \Pi(\Lambda)$ positive (negative) crossing $\leftrightarrow$ $|a_i|$ is even (odd)

So $\chi((\mathcal{A}_\Lambda, \partial_\Lambda)) = tb(\Lambda) + 2$

$\uparrow$ from t & t^{-1}

Now pick stabilized Legendrian knots Λ_1, Λ_2 with $tb(\Lambda_1) \neq tb(\Lambda_2)$. Then $\chi((\mathcal{A}_{\Lambda_1}, \partial_{\Lambda_1})) \neq \chi((\mathcal{A}_{\Lambda_2}, \partial_{\Lambda_2}))$, so the C.E. DGAs are not stable tame isomorphic, despite being quasi-isomorphic. $\blacksquare$

<u>Prop</u>. If $LCH(\mathcal{A}_{\Lambda_1}, \partial_{\Lambda_1}) = LCH(\mathcal{A}_{\Lambda_2}, \partial_{\Lambda_2})$ is trivial & $tb(\Lambda_1) = tb(\Lambda_2)$, then $(\mathcal{A}_{\Lambda_1}, \partial_{\Lambda_1})$ and $(\mathcal{A}_{\Lambda_2}, \partial_{\Lambda_2})$ are stable tame isomorphic.

<u>Fact</u>: (Sivek) $\exists$ Legendrian knots with trivial LCH which cannot be destabilized.

<u>Prop</u> (Cornwell, Ng, Sivek) For each $m \geq 1$, there is a Legendrian knot of smooth type $P(3,-3,-3-m)$ whose C.E. DGA is stable tame isomorphic to the C.E. DGA of .

<u>Upshot of these properties</u>: The C.E. DGA holds on to more data than is present in the LCH, but cannot detect all distinctions between Legendrian knots.

Augmentations

In many homology theories, the ranks of the graded pieces of the homology (i.e., the Betti numbers) can tell us important geometric information.

In LCH, these ranks can be infinite $\ddot{\frown}$

e.g., if $|a|=0$, then $|a^n|=0$, for $n \geq 1$, so the 0-graded portion of $\mathcal{A}_\Lambda$ has infinite rank, and this can persist to homology.

One solution is to construct "finite dimensional approximations" of $(\mathcal{A}_\Lambda, \partial_\Lambda)$, and compute homology from there.

These approximations require the existence of an augmentation of $(\mathcal{A}_\Lambda, \partial_\Lambda)$.

Given a unital ring S, let $(S, 0)$ denote the DGA whose elements are those of S, and whose grading and differential are both trivial.

Fix a Legendrian Λ, along with an integer ρ which divides $2\mathrm{rot}(\Lambda)$. Then a ρ-graded augmentation of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ to S is a DGA chain map

$$\varepsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \to (S, 0),$$

where $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is treated as a DGA with $\mathbb{Z}_\rho$ grading (induced by the $\mathbb{Z}$ grading).

We will most commonly take $S = \mathbb{Z}_2$ and $\rho = 0$, so that

$$\varepsilon : (\mathcal{A}_\Lambda, \partial_\Lambda) \longrightarrow (\mathbb{Z}_2, 0)$$

is any chain map satisfying $\varepsilon(1) = 1$, $\varepsilon \circ \partial_\Lambda$, and sending elements of nonzero degree to 0.

Such ε only exist if $\mathrm{rot}(\Lambda) = 0$.