Infinite-dimensional Morse theory and the Chekanov-Eliashberg DGA (2 talks)

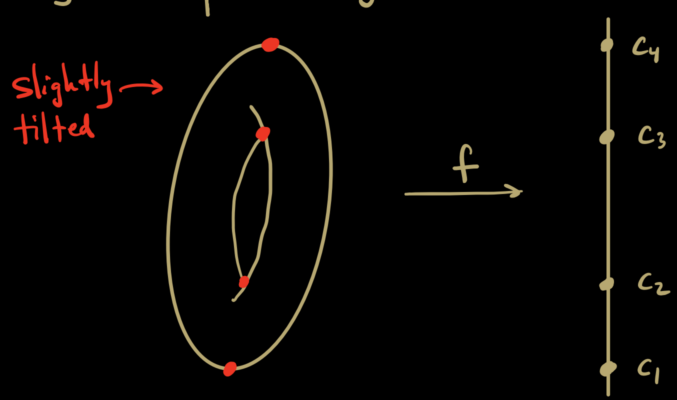
June 7

§1 A preview of the Morse DGA for surfaces

Setup: A nice function $f : X \to \mathbb{R}$ on a surface X

i.e., $f(x,y,z) = z$

Working example: height function on almost-upright torus



$\nabla f = \mathrm{proj}_{TX}\, \partial_z$

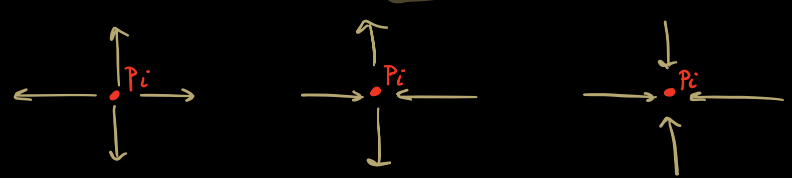
We construct a differential graded algebra $\mathcal{A}_X$ as follows:

- $\mathcal{A}_X$ is generated as an algebra by the critical points $p_1, p_2, \dots, p_m \in X$ of f:

  $\mathcal{A}_X := R\langle p_1, p_2, \dots, p_m\rangle.$

  some ring, probably $\mathbb{Z}_2$

- The grading is defined on generators according to the dimension of the stable set of the C.P.



$|p_i| = 2$ $\qquad$ $|p_i| = 1$ $\qquad$ $|p_i| = 0$

Gradings of products are defined by

$$|p_{i_1} p_{i_2} \cdots p_{i_k}| = |p_{i_1}| + |p_{i_2}| + \cdots + |p_{i_k}|.$$

- The differential counts gradient flowlines btwn critical points. Specifically:

$$\mathcal{M}(p_i, p_j) = \text{unstab}(p_i) \cap \text{stab}(p_j)$$

is a smooth mfld of dimension $|p_i| - |p_j|$ so we set

$$\partial p_i := \sum_{|p_j| = |p_i| - 1} \left(\# \mathcal{M}(p_i, p_j)/\sim \right) p_j ,$$

where $\sim$ denotes reparametrization, and extend to $\partial : \mathcal{A}_X \to \mathcal{A}_X$ algebraically. i.e., enforce linearity, Leibniz rule, etc.

<u>Note:</u> $\dim \mathcal{M}(p_i, p_j)/\sim \; = |p_i| - |p_j| - 1$, so the condition in our sum ensures that

$$\dim \mathcal{M}(p_i, p_j)/\sim \; = |p_i| - (|p_i| - 1) - 1 = 0,$$

allowing us to count the points in $\mathcal{M}(p_i, p_j)/\sim$.

Biggest takeaways for now:

- The Morse DGA for (X, f) is generated as an algebra by the C.P.s of f.
- The differential structure is obtained by counting gradient flowlines, where possible.
- The grading is defined to ensure that the differential has degree -1.

§2 The infinite-dimensional data

Just as Morse theory associates a DGA to a sm. mfld X, we want to associate a DGA to a Legendrian knot $\Lambda \subset \mathbb{R}^3$.

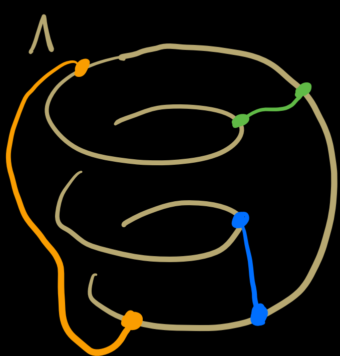
inf.-dim. "mfld" built from Λ

$$\Lambda \subset \mathbb{R}^3 \rightsquigarrow (X, \mathcal{A} : X \to \mathbb{R}) \rightsquigarrow \text{DGA}$$

"Morse theory"

Consider the set of parametrized chords in $\mathbb{R}^3$ with ends on Λ:

$$X = \{\gamma : [0,1] \to \mathbb{R}^3 \mid \gamma(0), \gamma(1) \in \Lambda\}.$$



This space is "infinite-dimensional" in the following sense: given any

$$V : [0,1] \to \mathbb{R}^3$$

with $V(i) \in T_{\gamma(i)}\Lambda$, we can perturb γ by "pushing in the direction of V" without leaving X.

Next, we need a nice function $X \to \mathbb{R}$. We'll use the action functional $\mathcal{A} : X \to \mathbb{R}$, defined by

$$\mathcal{A}(\gamma) := \int_\gamma dz - y\,dx.$$

We won't motivate this choice, except to say that physicists care about this functional.

Recall that if $\gamma(t) = (x(t), y(t), z(t))$, then
$$\int_\gamma dz - y\,dx = \int_0^1 (\dot{z}(t) - \dot{y}(t)\,x(t))\,dt.$$

§3 The critical points of $\mathcal{A}: X \to \mathbb{R}$

To do Morse theory with $(X, \mathcal{A})$, we need to identify the chords $\gamma \in X$ which satisfy $d\mathcal{A}_\gamma(V) = 0$ for every perturbation V. These will be the "critical points" which generate our DGA as an algebra.

I'll include a rough calculation here, but maybe not present it. The possibly-skipped material is in orange.

Consider $\gamma \in X$ and $V \in T_\gamma X$, and write
$$\gamma(t) = (x(t), y(t), z(t))$$
$$V(t) = (v_1(t), v_2(t), v_3(t)).$$
For any $\varepsilon > 0$,
$$\mathcal{A}(\gamma + \varepsilon V) = \int_0^1 \Big[(\dot{z}(t) + \varepsilon\,\dot{v}_3(t)) - (y(t) + \varepsilon\,v_2(t))(\dot{x}(t) + \varepsilon\,\dot{v}_1(t))\Big]\,dt$$
$$= \int_0^1 \big[\dot{z}(t) - y(t)\dot{x}(t)\big]\,dt + \varepsilon \int_0^1 \big[\dot{v}_3(t) - v_2(t)\dot{x}(t) - y(t)\dot{v}_1(t)\big]\,dt$$
$$- \varepsilon^2 \int_0^1 v_2(t)\,\dot{v}_1(t)\,dt$$
$$= \mathcal{A}(\gamma) + \varepsilon \int_0^1 \big[\dot{v}_3(t) - v_2(t)\,\dot{x}(t) - y(t)\,\dot{v}_1(t)\big]\,dt + \mathcal{O}(\varepsilon^2).$$

So $d\mathcal{A}_\gamma(V) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\mathcal{A}(\gamma+\varepsilon V) - \mathcal{A}(\gamma)\right]$

$$= \int_0^1 \left[\dot{V}_3(t) - V_2(t)\dot{x}(t) - y(t)\dot{V}_1(t)\right] dt.$$

Integration by parts yields

$$d\mathcal{A}_\gamma(V) = \left(V_3(1) - V_3(0)\right) - \int_0^1 V_2(t)\dot{x}(t)\,dt$$

$$- \left[\left(y(1)V_1(1) - y(0)V_1(0)\right) - \int_0^1 V_1(t)\dot{y}(t)\,dt\right].$$

Because $V(0)$ & $V(1)$ are tangent to Λ at $\gamma(0)$ and $\gamma(1)$, respectively, we have

$$V_3(1) - y(1)V_1(1) = 0 \quad \& \quad V_3(0) - y(0)V_1(0) = 0.$$

So

$$d\mathcal{A}_\gamma(V) = \int_0^1 \left[V_1(t)\dot{y}(t) - V_2(t)\dot{x}(t)\right] dt$$

$$= \int_0^1 \langle V_1(t), V_2(t)\rangle \cdot \langle \dot{y}(t), -\dot{x}(t)\rangle\, dt.$$

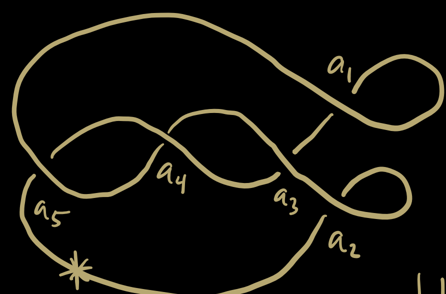
To have $d\mathcal{A}_\gamma(V) = 0$ for <u>all</u> V requires $\dot{x} \equiv \dot{y} \equiv 0$.

So the critical points of $\mathcal{A}$ are those chords $\gamma \in X$ satisfying $\gamma(t) = (x_0, y_0, z(t))$, for some function $z(t)$ and some constants x_0, y_0.

We call these Reeb chords.

§4 The graded algebra in the Lagrangian projection

We determined above that the algebra $\mathcal{A}_\Lambda$ associated to an oriented Legendrian knot $\Lambda \subset \mathbb{R}^3$ should be generated by Reeb chords with ends on Λ. These are easy to identify in $\pi(\Lambda)$: they correspond to double points. Up to a small perturbation of $\Lambda \subset \mathbb{R}^3$, we may assume that these are the only singularities of $\pi(\Lambda)$.



We'll include one more generator, t corresponding to a basepoint $* \in \pi(\Lambda)$ which we must choose. So the input data is: generic $\Lambda \subset \mathbb{R}^3$, plus a basepoint. We'll (kind of) address $*$ later.

<u>Def</u>. If $\pi(\Lambda)$ has transverse double points $a_1, a_2, \dots, a_n \in \pi(\Lambda)$ and basepoint $* \in \Lambda$, then $\mathcal{A}_\Lambda$ is the associative, noncommutative, unital algebra over $\mathbb{Z}$ generated by $a_1, \dots, a_n, t^{\pm 1}$, with relation $t \cdot t^{-1} = t^{-1} \cdot t = 1$:

$$\mathcal{A}_\Lambda := \mathbb{Z}\langle a_1, a_2, \dots, a_n, t^{\pm 1} \mid t t^{-1} = t^{-1} t = 1 \rangle.$$

<u>Rmk</u>. We generate $\mathcal{A}_\Lambda$ *as an algebra* by $a_1, a_2, \dots, a_n, t^{\pm 1}$. But $\mathcal{A}_\Lambda$ is infinitely generated as a module over $\mathbb{Z}$: its generators are words in the letters $a_1, \dots, a_n, t^{\pm 1}$.

§5 The grading on A_Λ

The grading we will define is difficult to motivate, except to say that it should help us measure the dimension of a certain moduli space.

e.g., we want the collection of "gradient flowlines" from a_i to a_j to be a mfld of dimension $|a_i| - |a_j|$.

We'll define the grading for each generator $a_1, \dots, a_n, t^{\pm 1}$. The grading of a product is then the sum of the gradings. (Recall that we only need to define the grading for the module generators.)

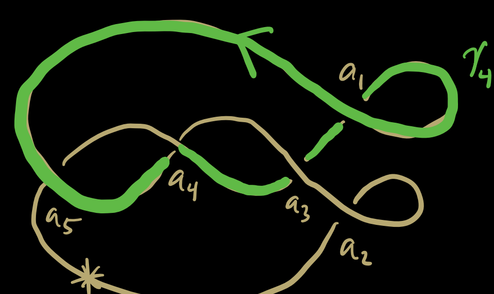
For $t, t^{-1} \in A_\Lambda$ we have

$$|t| := -2r(\Lambda) \quad ; \quad |t^{-1}| := 2r(\Lambda).$$

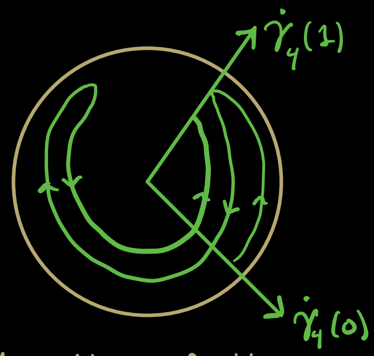
Defining $|a_i|$ is slightly weirder.

Given $a_i \in \pi(\Lambda)$, let γ_i be the unique path in $\pi(\Lambda)$ from the overcrossing of a_i to the undercrossing of a_i which avoids $*$. (No sharp turns.)

Let us assume that the strands which cross at a_i do so at a right angle. (This can be done by a generic perturbation.)



Then the vector $\dot{\gamma}_i$ will rotate through a net angle which is an odd multiple of $\pi/2$. We define $\mathrm{rot}(\gamma_i)$ to be this net angle, divided by 2π, so that $\mathrm{rot}(\gamma_i)$ is an odd multiple of $1/4$.



Then we define

$$\boxed{|a_i| := 2\,\mathrm{rot}(\gamma_i) - 1/2.}$$

In the above example, $\mathrm{rot}(\gamma_4) = 1/4$, so $|a_4| = 0$. Convince yourself that $|a_1| = 1$.

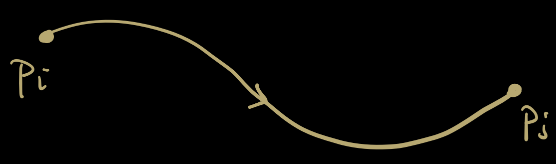
This definition probably seems strange, but is what's needed to make the moduli spaces behave correctly.

June 9

§6 Motivating the differential on $\mathcal{A}_\Lambda$

The differential in the Morse theory DGA comes from counting gradient flowlines between critical points, so we need to think about what this means.

In finite dimensions, when our C.P.s were points, we counted paths parallel to the gradient:



$\varphi : (-\infty, \infty) \longrightarrow X$

We didn't prove that $\partial \circ \partial = 0$ in Morse theory, but here's the basic idea, assuming that $R = \mathbb{Z}_2$.

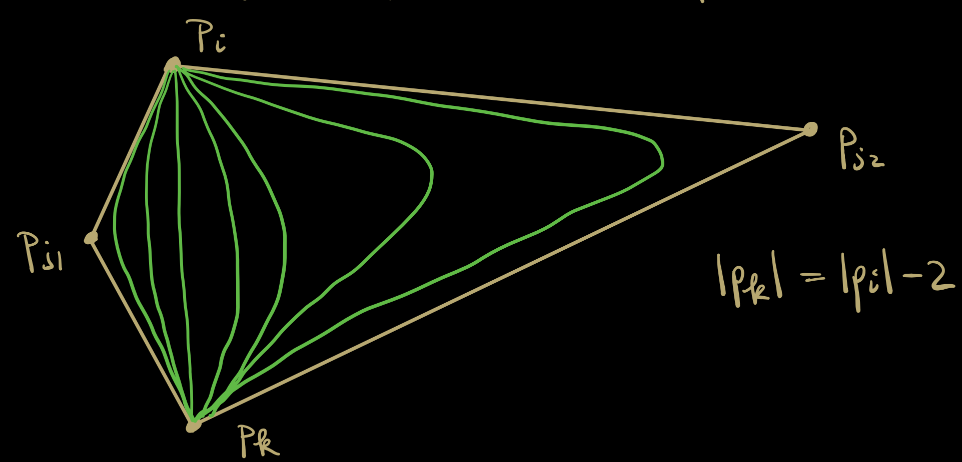
Recall that $\langle \partial p_i, p_j \rangle$ is a count of gradient flowlines from p_i to p_j, provided $|p_j| = |p_i| - 1$.
So

$$\partial(\partial p_i) = \partial\left(\sum_{|p_j| = |p_i| - 1} \langle \partial p_i, p_j \rangle p_j \right)$$

$$= \sum_{|p_j| = |p_i| - 1} \langle \partial p_i, p_j \rangle \, \partial p_j$$

$$= \sum_{|p_j| = |p_i| - 1} \langle \partial p_i, p_j \rangle \sum_{|p_k| = |p_j| - 1} \langle \partial p_j, p_k \rangle p_k .$$

In words, $\langle \partial^2 p_i, p_k \rangle$ counts the <u>broken</u> gradient flowlines $p_i \to p_j \to p_k$, with p_j arbitrary.



Notice that $\mathcal{M}(p_i, p_k)/\sim$ is a mfld of dim.

$$|p_i| - |p_k| - 1 = 2 - 1 = 1,$$

and its bdry consists of the "broken" flowlines we're counting.

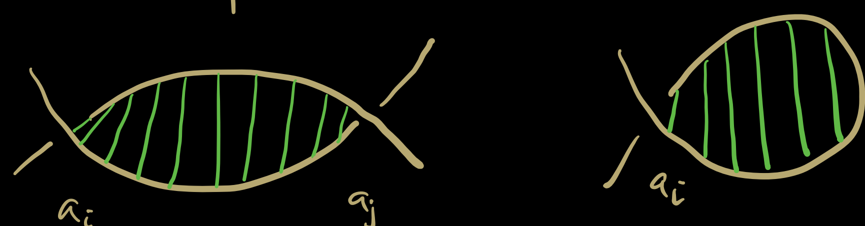
This is great, b/c the bdry of a 1D mfld consists of an even number of points, so #(broken flow lines $p_i \to p_k$) is even, and thus

$$\langle \partial^2 p_i, p_k \rangle \equiv 0 \pmod 2.$$

<u>Moral</u>: When defining ∂, we don't just need to understand gradient flow lines, but also how they degenerate.

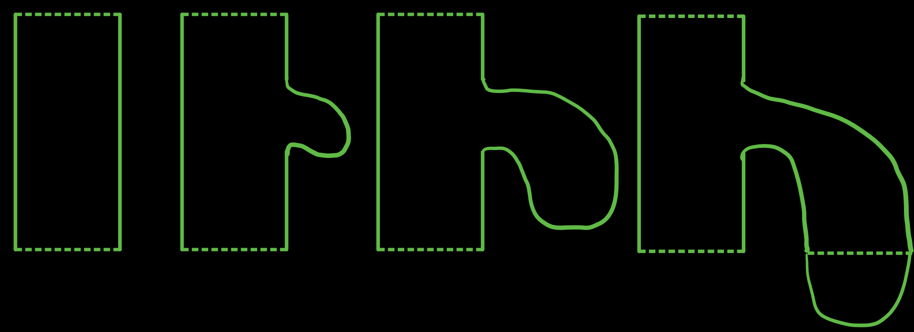
When our "critical points" are in fact chords, understanding these degenerations becomes MUCH more complicated.

e.g., a gradient flow line in this setting could be a strip $a_i \to a_j$

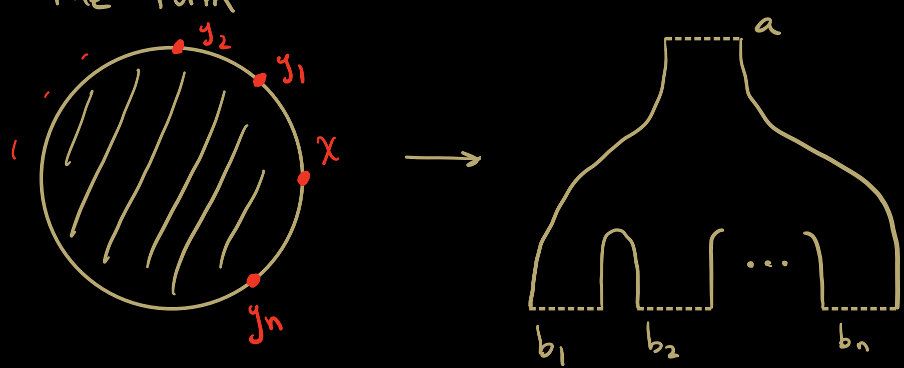


But there could also be flow lines which go nowhere. Laid flat:

A strip could then degenerate by sprouting one of these dead ends



Upshot: We won't win by just counting strips. Instead, we have to count flowlines of the form

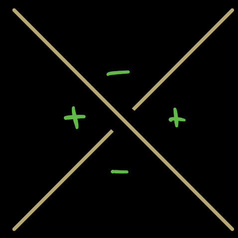


At this point, we'll stop trying to understand the differential from this geometric / analytic perspective, and just declare a combinatorial description.

§7 The differential on $\mathcal{A}_\Lambda$ in $\pi(\Lambda)$

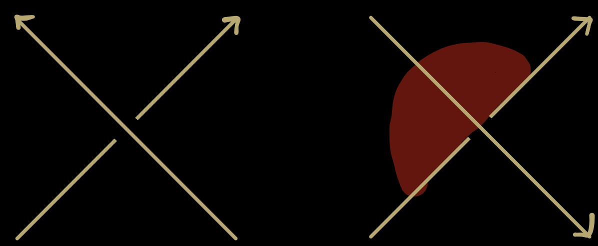
Computing ∂a will require that we count certain discs with boundary in $\pi(\Lambda)$, and identifying these discs requires that we first decorate each crossing of $\pi(\Lambda)$ with Reeb signs and orientation signs.

Reeb signs:



quadrant Q is positive if ∂Q, counterclockwise, moves from understrand to overstrand

Orientation signs:



negative crossing $\rightarrow$ all orientation signs are positive

positive crossing $\rightarrow$ shaded quadrants have negative orientation sign

<u>Notation</u>: $D^2_n = D^2 - \{x, y_1, \ldots, y_n\} \subset \mathbb{R}^2$, with punctures $x, y_1, \ldots, y_n \in \partial D^2$ counterclockwise.

Given intersection points $a, b_1, \ldots, b_n \in \Pi(\Lambda)$, we define a set

$$\Delta(a; b_1, \ldots, b_n) := \left\{ u : (D^2_n, \partial D^2_n) \rightarrow (\mathbb{R}^2_{xy}, \Pi(\Lambda)) \,\middle|\, \star \right\} / \sim,$$

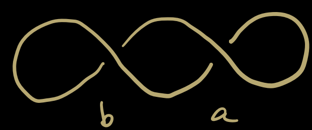
where $\sim$ denotes reparametrization (so we're only counting <u>images</u> of discs), and $\star$ denotes the following conditions:

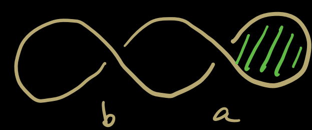
1. u is an immersion (i.e., it doesn't have to be injective, but no "pinch points");

② $\lim\limits_{p \to x} u(p) = a$, with a nbhd of x mapped to a quadrant of a with $+$ Reeb sign;

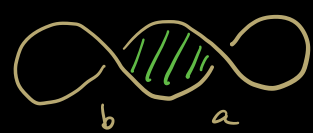
③ $\lim\limits_{p \to y_i} u(p) = b_i$, with a nbhd of y_i mapped to a quadrant of a with $-$ Reeb sign, for $1 \le i \le n$.

Ex





$\in \Delta(a)$ Any others?



$\notin \Delta(a; b)$

Very important fact:

If $\Delta(a; b_1, \dots, b_n) \neq \emptyset$, then

$$|a| - \sum_{i=1}^{n} |b_i| = 1.$$

So there's hope of defining ∂a to satisfy something like

$$\langle \partial a, b_1 b_2 \cdots b_n \rangle \overset{?}{=} |\Delta(a; b_1, \dots, b_n)|.$$

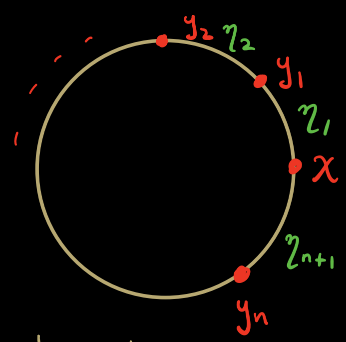
However, the basepoint and orientation signs require a bit more attention first.

Words

Each $u \in \Delta(a; b_1, \ldots, b_n)$ has an associated word

$$w(u) \in \mathbb{Z}\langle b_1, \ldots, b_n, t, t^{-1} \rangle.$$

Notice that ∂u consists of $n+1$ arcs $\eta_1, \ldots, \eta_{n+1}$ in $\Pi(\Lambda)$:



Let $k(\eta_i)$ be the signed count of the number of times η_i crosses $*$, and let $t(\eta_i) = t^{k(\eta_i)}$.

Then $w(u) := t(\eta_1)\, b_1\, t(\eta_2)\, b_2 \cdots t(\eta_n)\, b_n\, t(\eta_{n+1})$.

Signs: At each puncture $y_i \in D^2$, let $\varepsilon(b_i)$ be the orientation sign of the quadrant covered by u. Then

$$\varepsilon(u) := \prod_{i=1}^{n} \varepsilon(b_i).$$

At last, we can define ∂_Λ. For a double point a, we have

$$\partial a := \sum_{u \in \Delta(a; b_1, \ldots, b_n)} \varepsilon(u)\, w(u) \in \mathcal{A}_\Lambda,$$

where $n \geq 0$ and $b_1, \ldots, b_n$ can be any collection of double points in $\Pi(\Lambda)$ (with repetition allowed).

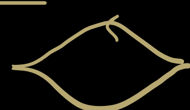
Fact: This is a finite sum, because gradient flow lines decrease action.

For t & t^{-1} we define $\partial_\Lambda t = 0$ & $\partial_\Lambda t^{-1} = 0$.
We extend to the module generators of $\mathcal{A}_\Lambda$ by the Leibniz rule:

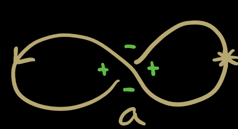
$$\partial_\Lambda(w_1 w_2) = (\partial_\Lambda w_1) w_2 + (-1)^{|w_1|} w_1 (\partial_\Lambda w_2),$$

and extend to arbitrary elements of $\mathcal{A}_\Lambda$ by linearity.

§8 Examples

① $\Lambda =$  $\rightsquigarrow$ 

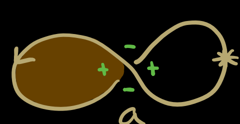
$\mathcal{A}_\Lambda = \mathbb{Z}\langle a, t, t^{-1}\rangle$
$|a| = 1, \; |t| = |t^{-1}| = 0$

decorations: 

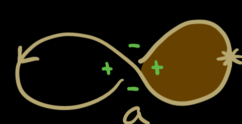
(negative crossing → pos. or'n signs)

$\partial_\Lambda t = 0, \; \partial_\Lambda t^{-1} = 0$

$\Delta(a) = \{u_1, u_2\}$



u_1



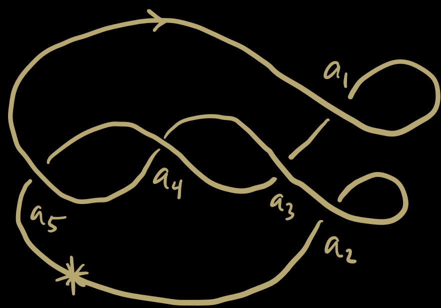
u_2

$W(u_1) = 1$, $W(u_2) = t^{-1}$

$\varepsilon(u_1) = 1$, $\varepsilon(u_2) = 1$

$$\therefore \partial_\Lambda a = 1 + t^{-1}$$

(2) $\Lambda =$

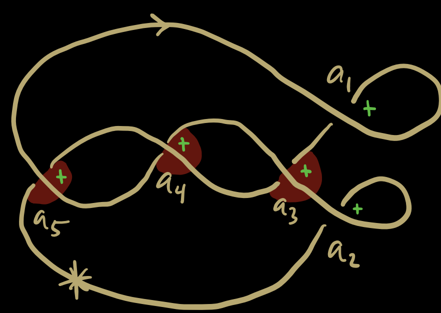


$|a_1| = |a_2| = 1$

$|a_3| = |a_4| = |a_5| = 0$

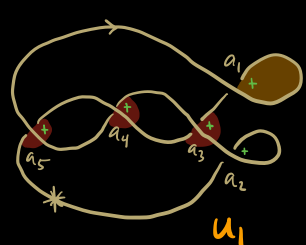
$|t| = |t^{-1}| = 0$

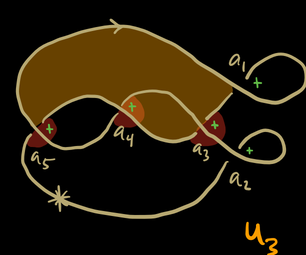
decorations:

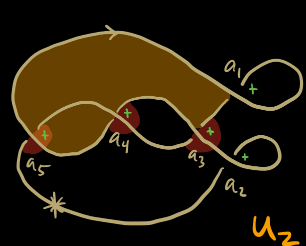


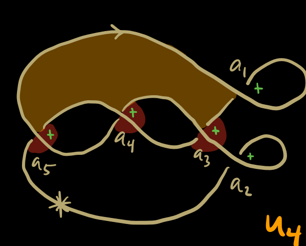
$\partial_\Lambda t = 0$, $\partial_\Lambda t^{-1}$

Let's compute $\partial_\Lambda a_1$. There are four discs which contribute to $\partial_\Lambda a_1$.









$u_1 \in \Delta(a)$, $u_2 \in \Delta(a_1; a_3)$, $u_3 \in \Delta(a_1; a_5)$,
$u_4 \in \Delta(a_1; a_5, a_4, a_3)$

$w(u_1) = 1$, $w(u_2) = a_1 a_3$, $w(u_3) = a_1 a_5$, $w(u_4) = a_5 a_4 a_3$
$\varepsilon(u_1) = \varepsilon(u_2) = \varepsilon(u_3) = \varepsilon(u_4) = 1$.

So $\partial_\Lambda a_1 = 1 + a_1 a_3 + a_1 a_5 + a_5 a_4 a_3$.

Time permitting, we'll also compute

$$\partial_\Lambda a_2 = 1 - a_3 t - a_5 t - a_3 a_4 a_5 t.$$