1 May 31 Exercises

1.1 Introduction to LCH

1. Consider the parametrization $(x(t), z(t)) \in \mathbb{R}^2_{x,z}$ given by

$$(x(t), z(t)) = (t^2, t^3), \quad t \in \mathbb{R}.$$

Find a Legendrian parametrization $(x(t), y(t), z(t)) \in \mathbb{R}^3_{x,y,z}$ whose front projection is given by the above. That is, your parametrization should satisfy $z'(t) - y(t)x'(t) = 0$. (But it won't be a knot.)
Answer: $(t^2, \frac{3}{2}t, t^3)$.

1.2 Differential forms on $\mathbb{R}^3$

1. Consider the vectors $\mathbf{v} = (7, 0, 5)$ and $\mathbf{w} = (3, 10, 5)$ in $\mathbb{R}^3$. Compute $\eta(\mathbf{v}, \mathbf{w})$, where

$$\eta = 3\, dy \wedge dz - 4\, dz \wedge dx + 5\, dx \wedge dy.$$

Answer: 280

2. Show that if $f : \mathbb{R}^3 \to \mathbb{R}$ is any smooth function on $\mathbb{R}^3$ and $\mathbf{v} \in \mathbb{R}^3$ is an arbitrary vector, then

$$\nabla f \cdot \mathbf{v} = \frac{\partial f}{\partial x}\, dx(\mathbf{v}) + \frac{\partial f}{\partial y}\, dy(\mathbf{v}) + \frac{\partial f}{\partial z}\, dz(\mathbf{v}).$$

Here ∇f denotes the gradient of f. The 1-form

$$df := \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

is often called the **total differential** of f, and this exercise shows that df is dual to ∇f.

3. Let $\mathcal{C}$ be the curve in $\mathbb{R}^3$ parametrized by $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$, $0 \le t \le \pi$, and let $\alpha = dz - y\, dx$. Evaluate $\int_{\mathcal{C}} \alpha$.
Answer: 2π

4. Assuming that λ and η are 1-forms on $\mathbb{R}^3$, verify that

$$\mathbf{X}_\lambda \times \mathbf{X}_\eta = \mathbf{X}_{\lambda \wedge \eta}.$$

This completes a computation from the lecture.

5. Suppose we have a 1-form λ on $\mathbb{R}^3$ with $X_\lambda = \langle F_1, F_2, F_3 \rangle$ and consider the 2-form defined by

$$d\lambda := dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz.$$

Here the symbol $d\lambda$ is (for now) meaningless, except as we've just defined it. Each of dF_1, dF_2, and dF_3 is a total differential, as defined above. Show that

$$d\lambda = \mathrm{curl}_x(\mathbf{X}_\lambda)\, dy \wedge dz + \mathrm{curl}_y(\mathbf{X}_\lambda)\, dz \wedge dx + \mathrm{curl}_z(\mathbf{X}_\lambda)\, dx \wedge dy,$$

where $\mathrm{curl}_x(\mathbf{X}_\lambda)$ denotes the x-component of the vector field $\mathrm{curl}(\mathbf{X}_\lambda)$, and similarly for the other components.

6. (Challenge) Consider a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, and let $\mathcal{S} \subset \mathbb{R}^3$ be the surface defined by

$$\mathcal{S} := f^{-1}(c) := \{(x, y, z) : f(x, y, z) = c\},$$

for some $c \in \mathbb{R}$. Assume that c is a *regular value* for f, meaning that if $f(p) = c$, then $\nabla f(p) \neq \mathbf{0}$. (This ensures that $\mathcal{S}$ is a *smooth* surface.) Prove that if $\mathbf{v} \in \mathbb{R}^3$ is tangent to $\mathcal{S}$ at a point $p \in \mathcal{S}$, then $df(\mathbf{v}) = 0$.
Hint: Use the duality between df and ∇f, as well as a fact about how ∇f relates to the level sets of f.

2 June 1 Exercises

2.1 Knots

1. Compute the Alexander polynomials of the Stevedore Knot (6_1) and the Figure Eight knot(4_1) and show that they are not isotopic to the unknot.
2. Use Reidemeister moves to prove that the Figure Eight knot is isotopic to its mirror.
3. Use Reidemeister moves to show that interchanging the two components of the Whitehead link gives an isotopic link.
4. Prove that the Alexander polynomial is invariant under the R2 and R3 moves.

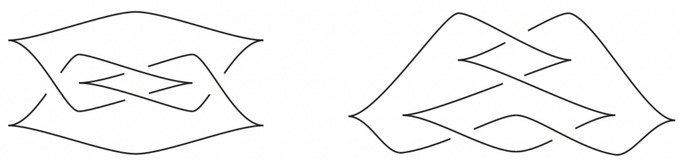
2.2 Differential forms on $\mathbb{R}^3$

1. Prove the product rule for total differentials: $d(fg) = g\,df + f\,dg$, for any smooth functions $f, g : \mathbb{R}^3 \to \mathbb{R}$.
2. Verify that exterior differentiation satisfies the **Leibniz rule**:
$$d(\lambda \wedge \eta) = d\lambda \wedge \eta + (-1)^{\deg(\lambda)} \lambda \wedge d\eta,$$
for any k-form λ and any ℓ-form η. (You should only have to check a few cases.)
3. Prove that if λ is a k-form, then $d(d\lambda) = 0$. This is sometimes written as $d^2 = 0$. (Since we're working in $\mathbb{R}^3$, the cases $k = 2$ and $k = 3$ should basically be free.)
Hint: For the case $k = 1$, use the Leibniz rule.
4. Let $\alpha = dz - y\,dx$.
   (a) Check that the differential equation $\alpha(\dot{\gamma}(t)) = 0$ is equivalent to $z'(t) - y(t)x'(t) = 0$.
   (b) Compute $\alpha \wedge d\alpha$.
5. Check that $dz + r^2\,d\theta$ is a contact form on $\mathbb{R}^3$, where (r, θ, z) are cylindrical coordinates on $\mathbb{R}^3$. That is, $x = r\cos\theta$ and $y = r\sin\theta$.
Hint: You'll need to compute $dx \wedge dy$ in terms of dr and $d\theta$.
6. Show that $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ is a symplectic form on $\mathbb{R}^4$. This is called the **standard symplectic form** on $\mathbb{R}^4$.
7. (Challenge) Let $\mathbb{R}^4$ have coordinates (s, x, y, z). Given that α is a contact form on $\mathbb{R}^3$ (with coordinates (x, y, z)), show that $\omega = d(e^s\alpha)$ is a symplectic form on $\mathbb{R}^4$.
8. Let $\mathcal{S} \subset \mathbb{R}^3$ be the surface parametrized by
$$F(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad 0 \le u \le \pi, 0 \le v \le 2\pi.$$
Show that
$$\int_{\mathcal{S}} \nu = 4\pi,$$
where
$$\nu = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$
Can you identify the surface $\mathcal{S}$?
9. Let $\mathcal{B}$ be the ball of radius 1 centered at the origin in $\mathbb{R}^3$, and compute
$$\int_{\mathcal{B}} d\nu,$$
where ν is the 2-form identified in the previous exercise.
Hint: You shouldn't need to evaluate a triple integral if you remember the volume of a ball.

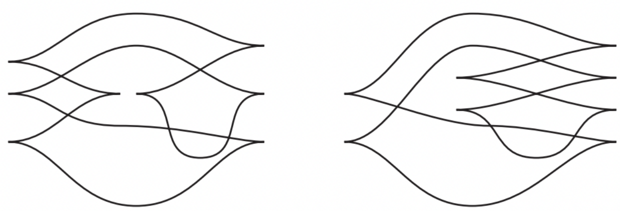
3 June 2 Exercises

3.1 Knots

1. Find a Seifert surface for the Figure Eight knot and prove that it is minimal genus.
2. Find a Legendrian isotopy, using Legendrian Reidemeister moves, between these two front projections of the Figure Eight knot.



3. Show that the Thurston-Bennequin invariant, as defined using the front projection, is an invariant of Legendrian knots.
4. Find a diagram for the trefoil for which Seifert's algorithm gives a genus 2 surface. (think about how many disks and bands would be necessary)
5. Prove that the knots pictured are isotopic as smooth knots, and have the same Thurston-Bennequin and Rotation numbers.



3.2 Differential graded algebras

1. Show that any abelian group G can be made into a $\mathbb{Z}$-module as follows: first, write the group operation of G as $+$. Then, for any $n \in \mathbb{Z}$ and $g \in G$, define

$$ng = \begin{cases} g + g + \cdots + g \ (n \text{ times}), & n > 0 \\ 0_G, & n = 0 \\ -g - g - \cdots - g \ (-n \text{ times}), & n < 0 \end{cases},$$

where 0_G is the additive identity of G.

2. Let A be an associative algebra over a commutative, unital ring R. Define the *center* C_A of A as follows:

$$C_A := \{c \in A \,|\, a \cdot c = c \cdot a, \text{ for all } a \in A\}.$$

Verify that C_A is an algebra over R.

3. Let X be (the boundary of) a cube in $\mathbb{R}^3$. For concreteness, you can take X to be the boundary of $[0,1] \times [0,1] \times [0,1]$, but this isn't necessary. Next, let $\mathcal{C}_0$ be the $\mathbb{Z}$-module freely generated by the vertices $v_1, v_2, \ldots, v_8$ of X. That is, $\mathcal{C}_0$ consists of formal linear combinations of $v_1, v_2, \ldots, v_8$, with coefficients coming from $\mathbb{Z}$. Similarly, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the free $\mathbb{Z}$-modules freely generated by the edges and faces of X, respectively.

(a) For $k = 1$ and $k = 2$, define a map $\partial_k : \mathcal{C}_k \to \mathcal{C}_{k-1}$ which corresponds to taking a boundary. For instance, the boundary of a face should be a linear combination of four edges. To define ∂_k, you will need to choose an orientation for each edge and face. (It makes sense to orient each face counterclockwise when viewed from outside the cube.) By "define a map," we just mean that you should specify what it does to each generator of $\mathcal{C}_k$.

(b) Produce a matrix representative for each of ∂_1 and ∂_2. Your matrix for ∂_2 should be 12×6, and your matrix for ∂_1 should be 8×12. Use these matrices to verify that $\partial_1 \circ \partial_2 = 0$.
Warning: Your matrix representatives will depend on your labeling of the cube; so there are many correct answers.

(c) The previous part shows that the image of ∂_2 is a subset of the kernel of ∂_1. (Make sure you believe this.) Show that these two $\mathbb{Z}$-submodules of $\mathcal{C}_1$ are in fact equal.
Hint: Identify bases for image(∂_2) *and* ker(∂_1).

(d) With what we've done so far, we could define $\mathcal{C}_\bullet$ to be the $\mathbb{Z}$-module freely generated by the vertices, edges, and faces of X, and $\partial : \mathcal{C}_\bullet \to \mathcal{C}_\bullet$ would be a linear map which decreases grading by 1 and squares to zero. (We can just define the boundary of any vertex to be zero.) Why is $\mathcal{C}_\bullet$ not a DGA?

4. Let $\mathcal{R} = \mathbb{R}^2 - \{(0,0)\}$ and consider the 1-form $\lambda \in \Omega^1(\mathcal{R})$ defined by

$$\lambda = -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy.$$

(a) Verify that $d\lambda = 0$.

(b) Next, we want to show that λ is *not* the total differential of some function. We'll do this in steps.

i. Let $\mathcal{C}$ be parametrized by $\mathbf{r}(t) = (\cos 2\pi t, \sin 2\pi t)$, $0 \le t \le 1$. Compute $\int_{\mathcal{C}} \lambda$.

ii. (Challenge) With the same $\mathcal{C}$, show that if $f : \mathcal{R} \to \mathbb{R}$ is a smooth function, then

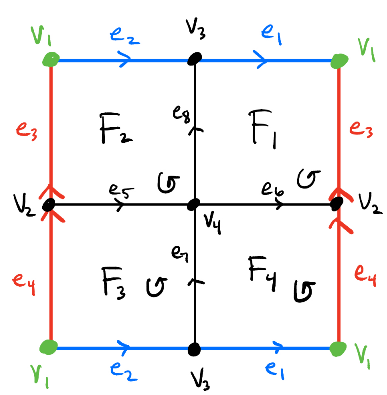
$$\int_{\mathcal{C}} df = 0.$$

Hint: At some point you'll need the following version of the chain rule: $\frac{d}{dt}(f(\mathbf{r}(t))) = \nabla f \cdot \mathbf{r}'(t)$.

iii. Use the previous two parts to conclude that $\lambda \neq df$ for any function $f : \mathcal{R} \to \mathbb{R}$.

This problem demonstrates that image(d) can be a strict subset of ker(d); the difference between these two sets is determined by the topology of $\mathcal{R}$.

5. The following is a representation of the torus T^2:



We obtain a surface in $\mathbb{R}^3$ by identifying the blue horizontal edges in a way that respects their orientations, and doing the same for the red vertical edges. Just as we did for the cube in problem 3, we break T^2 into faces; these faces have edge boundaries, and the edges have vertex boundaries. The orientations are as labeled. (Notice the repeated labels! Those are intentional!) Repeat as much of problem 3 as you can, with T^2 playing the role previously played by X.

4 June 3 Exercises

1. For the Legendrian trefoil presented in the lecture, compute $\partial_\Lambda(a_k)$, for $k = 2, 3, 4, 5$.
2. In problem 5 of the knots section of yesterday's exercises, two Legendrian knots are given. Produce a Lagrangian projection for the knot on the left and then compute its Chekanov-Eliashberg DGA. (Orient the top strand left-to-right, and put the basepoint on the bottom strand.)
3. For the DGA you computed in the previous problem, set $t = 1$ and reduce all integers mod2 to produce a DGA over $\mathbb{Z}/2\mathbb{Z}$. Find all augmentations of this DGA. (This should look familiar!)

5 June 6 Exercises

1. Let Λ_1 and Λ_2 be the Chekanov-Eliashberg pair, the front projections of which are given in problem 5 of section 3.1 above. Compute $(\mathcal{A}_{\Lambda_i}, \partial_{\Lambda_i})$ from the front projection, for $i = 1, 2$. (We'll need to fix orientations and basepoints.)

2. For the DGAs computed in the previous problem, set $t = 1$ and reduce all coefficients modulo 2, so that we have DGAs over $\mathbb{Z}/2\mathbb{Z}$. Find all augmentations of these DGAs, and compute the corresponding linearized homologies. (This is a restatement of a problem from Friday.)

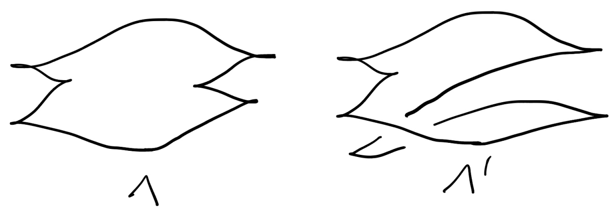
3. Describe how augmentations are helpful for computing invariants.

4. Let Λ_1 be the result of positively stabilizing the standard Legendrian unknot, and let Λ_2 be a Reeb pushoff of Λ_1. That is, Λ_2 is obtained from Λ_1 by a small translation in the z-direction. Compute the DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$, where $\Lambda = \Lambda_1 \cup \Lambda_2$. (Choose your Maslov potential to match on the two components of Λ, with "match" having the obvious meaning.)

6 June 7 Exercises

For the most part, today's problem session will focus on catching up on leftover exercises. But here's one new one:

1. Let Λ and Λ' be the Legendrian knots whose front projections are given:



Find a stable tame isomorphism $\Phi\colon (\mathcal{A}_\Lambda, \partial_\Lambda) \to (\mathcal{A}_{\Lambda'}, \partial_{\Lambda'})$. (Work with $\mathbb{F}_2$ coefficients.)