Surfaces bound by knots:

1) Seifert's algorithm
2) Ribbon knots and slice surfaces

1) e.g.

2) e.g.

Skeinoid: Knot (61).

2 disks (pink) + 1 band (blue) = disk

\[ X = \# \text{ disks} - \# \text{ bands} = 2 - 2g - 1 \quad (g: \text{ genus}) \]

Ribbon knot

Three self-intersections in surfaces can be pushed apart in 4 dimensions

Surface in \( S^3 \)

Surface in \( B^4 \) without self-intersection

\( \text{If knot can be described by } n \text{ disks and } n-1 \text{ bands, it is slice (also ribbon)} \)
$g(K) := \text{min genus of Seifert surface}$

$g_4(K) := \text{min genus of slice surface}$

- **Linking number and writhe**

\[ \begin{array}{ccc}
\text{+}&\text{x}&\text{-} \\
\end{array} \]

**Notation:** For an oriented knot diagram, signed sum of crossings. (not a knot invariant)

**E.g.**

\[ \text{writhe} = -3. \]

Linking number: given knots $K$ and $K'$,

\[ \text{lk}(K, K') = \text{signed count of all crossings where } K \text{ goes over } K' \]

(\text{or vice-versa})

**E.g.**

\[ \text{# crossings where } K' \text{ goes over } K : 5 \]

\[ \text{lk}(K, K') = +2 - 3 = -1 \]
Legendrian knots: image of $\Phi: \mathbb{S}^1 \to \mathbb{R}^3$ satisfies $z'(\theta) - y(\theta)x'(\theta) = 0$.

Front projections: $\Pi: (x, y, z) \to (x, z)$
- no vertical tangencies, no crossing.
- given a knot diagram in $\mathbb{R}^2$ without vertical tangencies, can choose parametrisation to get a well-defined Legendrian corresponding to it.

Legendrian Reidemeister moves:

- $R_1$

- $R_2$

- $R_3$

Theorem: Two front projections represent isotopic Legendrian knots $\iff$ they are related via regular homotopy and a sequence of above moves.
• Given a smooth knot, can obtain a Legendrian representative by:

\[ \rightarrow \quad \rightarrow \quad \times \rightarrow \quad \text{or} \]

\[ \leftarrow \quad \leftarrow \quad \leftarrow \]

• Invariants:

1) Thurston-Bennequin: \( \text{lk}(K, K') \)

\[ = \text{writhe}(K) - \frac{1}{2} \# \text{cups} \]

2) Rotation number: \# times tangent vector to interior of oriented leg of knots:

\[ \text{rot}(K) = \frac{1}{2} \left( \# \text{down cups} - \# \text{up cups} \right) \]
\( \text{Lagrangian projection} \quad (x, y, z) \rightarrow (x, y) \)

\( \rightarrow \) there is a weak Reidemeister-type theorem, no iff condition.

\( \text{Front} \)

\( \text{Lagrangian} \)

\( \langle \quad \rangle \rightarrow \infty \)

\( \text{Turning front projection to Lagrangian} \)

\( \rightarrow \quad \left\{ \right\} \)

\( \text{Double points of Lagrangian projection} \)

\( \text{Reeb chords} \)
\[ \text{Fillings of Legendrian : } L^1 \text{ surfaces in } B^4 \]

\[ K \]

\[ S^3 \]

\[ \Sigma \]

\[ B^4 \]

**Fact:** These surfaces minimise genus

\[ g(\Sigma) = g_4(K) \]

- Need not always exist

**Q:** Given a knot, how may Lagrangian fillings (up to some notion of isotopy) does it admit?

\[ LCH(K) \] gives information about this question as well.