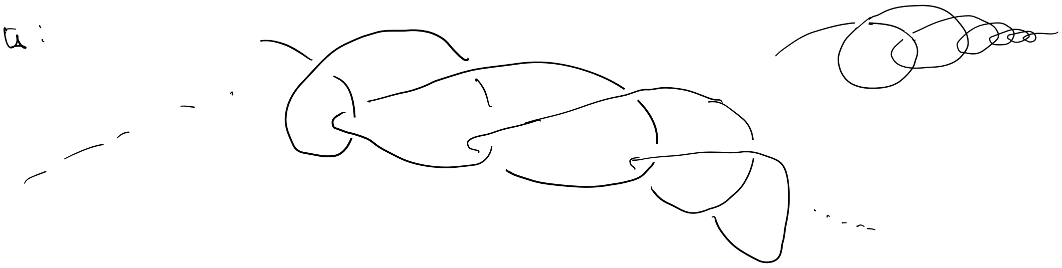
Knot: Image of $\varphi : S^1 \longrightarrow \mathbb{R}^3$

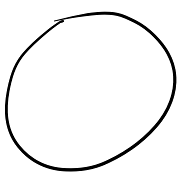
continuous, 1-1 to image, no "pathologies".

Wild knots:

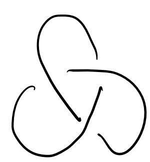
X



Tame knots.



Unknot



Trefoil

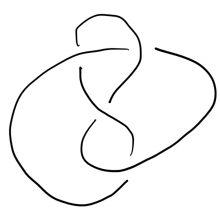
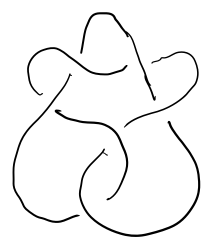
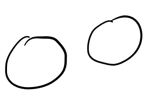


Figure Eight Knot

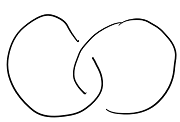


Stevedore Knot

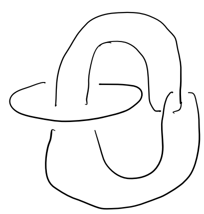
Links:



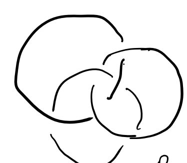
Unlink



Hopf link



Whitehead Link



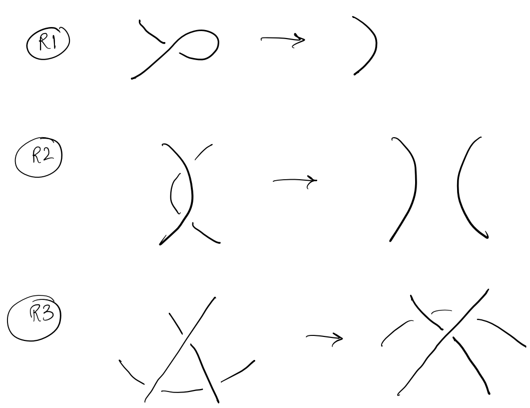
Borromean rings

Equivalence: If you can turn one diagram to
(idea) another without cutting, just by moving strands.

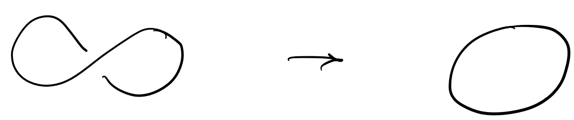
Formally: Two knots are equivalent if there is an ambient isotopy between them.

ie $F: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$, $F_0 = id.$, $F_1(K) = K'$, each F_t is a homeo.

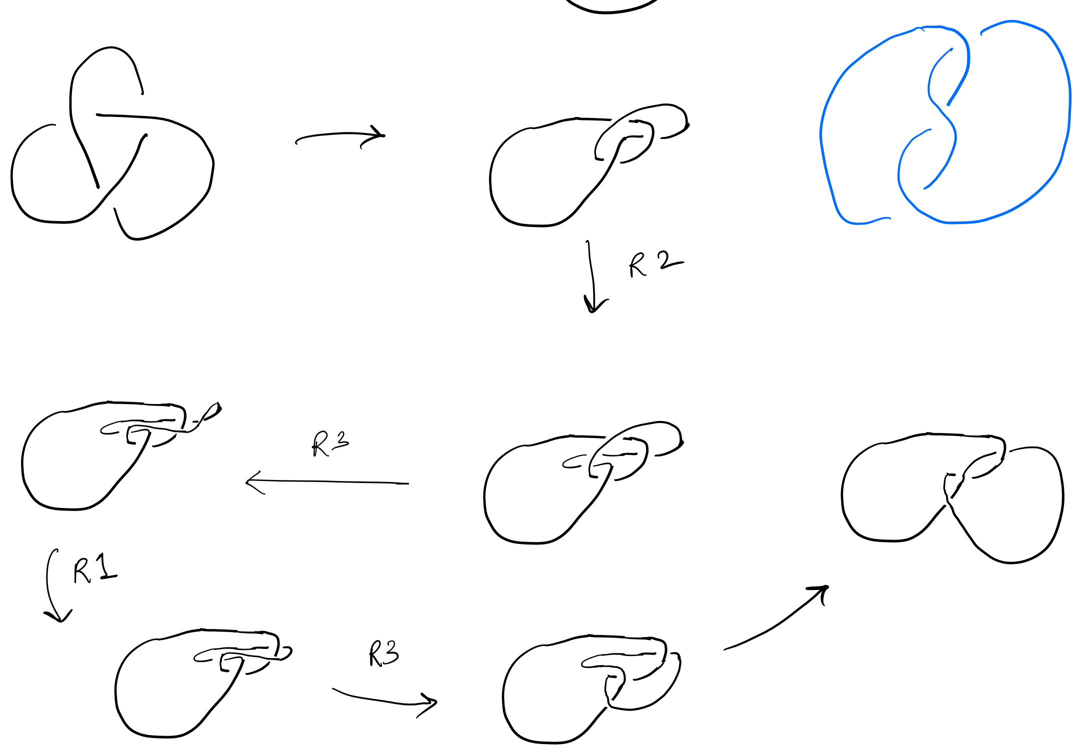
Reidemeister moves:



Example: 1)



2)



Exercise:

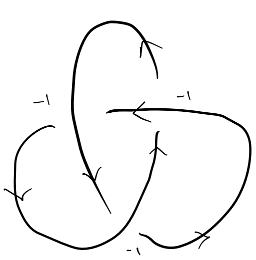
1) Interchanging components of Whitehead link.
2) Figure-Eight knot is isotopic to its mirror.

Mirror of a knot: Reverse all crossings.

Theorem: Two knots are equivalent $\iff$ they are connected by a sequence of planar ambient isotopies, and a finite series of Reidemeister moves or their inverses

- **Writhe:** # positive crossings − # negative crossings (of a diagram).

e.g.



writhe $= -3$

- **Knot invariants :**

1) Knot group : $\pi_1(\mathbb{R}^3 \setminus K)$

- every diagram gives a presentation called Wirtinger presentation.

Ex: What is $\pi_1(\mathbb{R}^3 \setminus U)$?

2) Alexander polynomial :

oriented diagram

n crossings $\longrightarrow$ n+2 regions

Incidence matrix

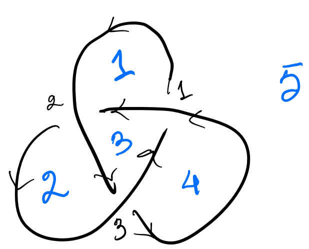
$$n\left\{\begin{array}{|c} \\ \\ \\ \end{array}\right. \quad \underbrace{\qquad\qquad\qquad}_{n+2}$$

- if region not adjacent to crossing, entry 0
- det. by location of region w.r.t. incoming undercrossing line
  - i) left before u.c. : $-t$
  - ii) right before u.c : 1
  - iii) left after u.c. : t
  - iv) right after u.c : -1

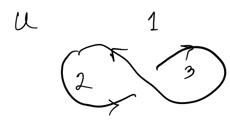
Remove two columns corr. to adjacent regions,

$$\Delta_K(t) := \det(\text{new } n\times n \text{ matrix})$$

e.g.



	1	2	3	4	5
1	t	0	$-t$		
2	1	t	$-t$	0	
3	0	1	$-t$		



	2
1	$-t$

$$\det \begin{vmatrix} t & 0 & -t \\ 1 & t & -t \\ 0 & 1 & -t \end{vmatrix}$$

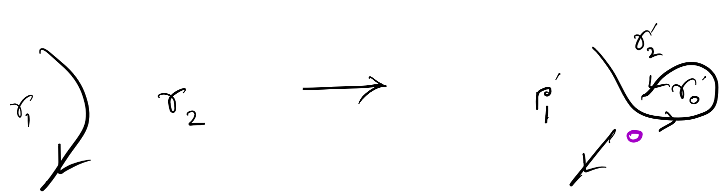
$$= t\left(-t^2+t\right) - t \cdot 1$$

$$= -t^3 + t^2 - t \simeq -t + 1 - \frac{1}{t}$$

- Exercise: Show that $\Delta_K(t)$ does not detect the unknot, i.e. $\exists$ knot K, not isotopic to U, such that $\Delta_K(t) = \pm t^{\pm n}$.

- Theorem: The Alexander polynomial is a knot invariant, up to multiplication by $\pm t^n$; $n \in \mathbb{Z}$.

Proof: RI:



$$
\begin{array}{c|ccc}
 & r_1 & r_2 & \cdots \\
\hline
1 & & & \\
2 & & M & \\
\vdots & & &
\end{array}
\qquad\qquad
\begin{array}{c|cccc}
 & r_0' & r_1' & r_2' & \cdots \\
\hline
0 & -t & * & * & \cdots \\
1 & 0 & & & \\
2 & 0 & & M & \\
\vdots & \vdots & & & \\
 & 0 & & &
\end{array}
$$

<u>Exercise</u>: Show invariance under R2 and R3.