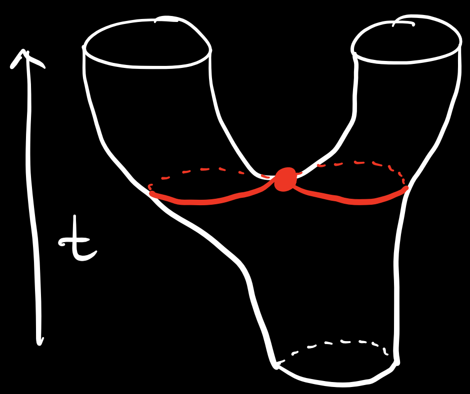
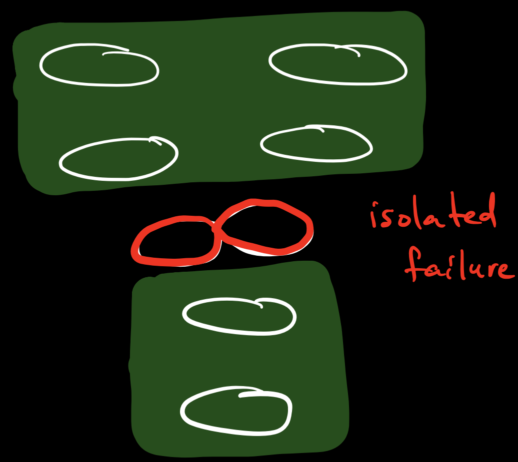
Geometric background on cobordisms





Interesting stuff happens at critical points.

If we have a cobordism

$$\mathcal{O}_1 \longrightarrow \mathcal{O}_2$$

and also some invariants $I(\mathcal{O}_1)$ & $I(\mathcal{O}_2)$, we want a relationship btwn $I(\mathcal{O}_1)$ & $I(\mathcal{O}_2)$ coming from the cobordism.

For us: nice objects are Leg. knots

invariant is Chekanov-Eliashberg DGA

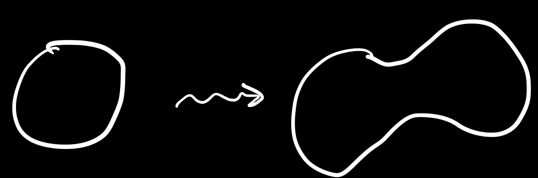
cobordisms are exact Lagrangian cobordism

What geometric conditions should our cobordisms have?

(Already have top. conditions: cobordism should be a surface with bdry, slices should be knots.)

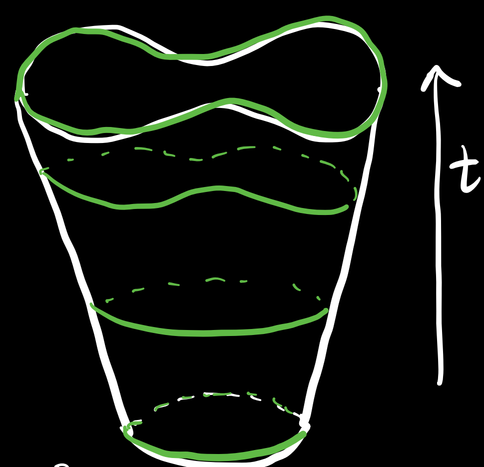
Most basic cobordism: cylinder over a knot (or Legendrian knot). (should induce identity on $I(\Theta_i)$)

Next most basic: "movie" of an isotopy.



Isotopy K_t, $t \in [0,1]$

$\rightsquigarrow C = \{(t, K_t) \mid t \in [0,1]\}$
$\subset \mathbb{R} \times \mathbb{R}^3$



is our cobordism.

Now say we have a Legendrian isotopy $\Lambda_t \subset (\mathbb{R}^3, \lambda_0 := dz - y\,dx)$.

should take kernel

Cobordisms should live in the symplectic space $(\mathbb{R}^4, \omega = d(e^t \lambda_0))$, where $\mathbb{R}^4$ has coords t, x, y, z.

Aside: We've built the "symplectization" of our contact manifold. Idea: contact mflds arise as constant-energy hypersurfaces in sympl. mflds.

"Watching the movie" of the isotopy should be a cobordism. So set

$$L := \{(t, \Lambda_t) \mid t \in [0,1]\} \subset \mathbb{R}^4_{t,x,y,z}.$$

Geometric conditions tell us how L interacts with ω.

$$\omega = d(e^t\lambda_0) = d(e^t)\wedge\lambda_0 + (-1)^{\deg(e^t)}\cdot e^t\wedge d\lambda_0$$

$$= d(e^t)\wedge\lambda_0 + e^t d\lambda_0$$

$$= e^t dt\wedge\lambda_0 + e^t d\lambda_0.$$

Let's consider $\omega|_L$. i.e., we plug into ω only vectors which are tangent to L.

- $d\lambda_0$ is independent of dt $\rightarrow$ plugging in ∂_t (vector in t-direction) gives zero.

$$e^t d\lambda_0|_L \equiv 0$$

- $e^t dt\wedge\lambda_0$. Plugging the Legendrian direction, tangent to Λ_t, kills this.

$$e^t dt\wedge\lambda_0|_L \equiv 0.$$

So $\boxed{\omega|_L \equiv 0.}$ We call surfaces in $(\mathbb{R}^4, \omega)$ <u>Lagrangian</u> if ω restricts to 0 on them.

In fact, not only does $d(e^t\lambda_0)$ vanish on L, $e^t\lambda_0$ itself vanishes on L:

- plugging ∂_t into $e^t\lambda_0$ gives 0;
- plugging in direction tangent to Λ_t also gives 0, since Λ_t is Leg.

So $e^t\lambda_0|_L \equiv 0$.

Remember: We intend to allow cobordisms to fail to be isotopies at isolated points.

We relax our geometric condition by requiring a Lagrangian cobordism to satisfy

$$e^t\lambda_0|_L = df,$$

for some $f: L \to \mathbb{R}$. This will allow the isolated failure we want.

Def. Let $\Lambda_+, \Lambda_- \subset \mathbb{R}^3$ be Legendrian links.

An <u>exact Lagrangian cobordism</u> from Λ_+ to Λ_- is a Lagrangian submanifold L of $(\mathbb{R}^4, d(e^t \lambda_0))$ such that

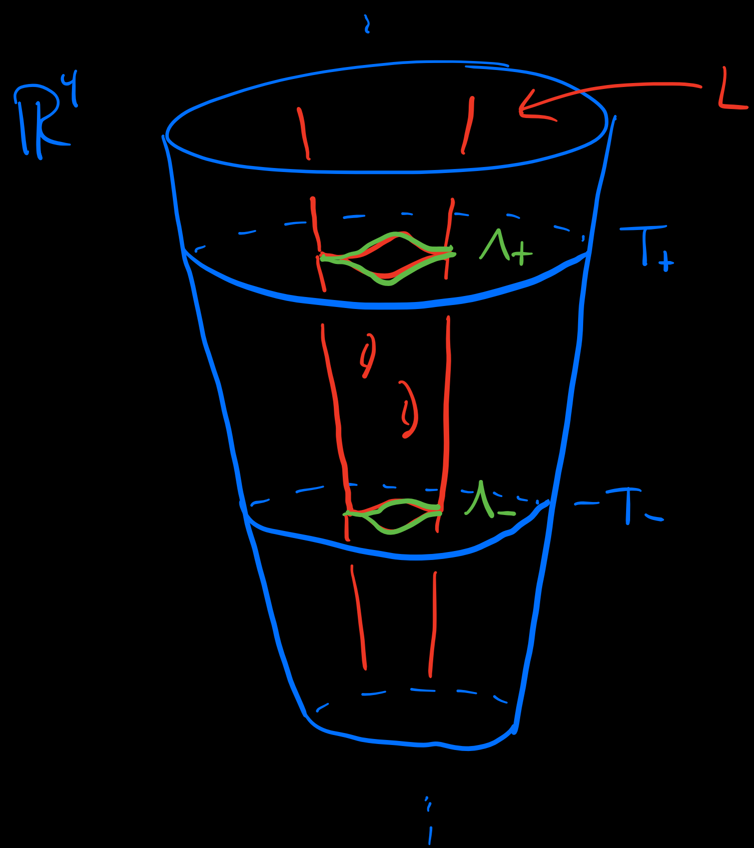
- for some $T_- >> 0$,

  $L \cap ((-\infty, -T_-) \times \mathbb{R}^3) \simeq (-\infty, -T_-) \times \Lambda_-$;

- for some $T_+ >> 0$,

  $L \cap ((T_+, \infty) \times \mathbb{R}^3) \simeq (T_+, \infty) \times \Lambda_+$;

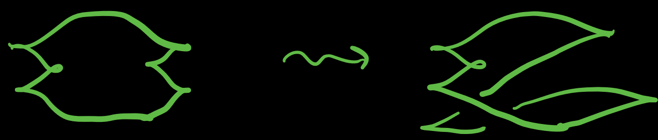
- $\exists$ function $f : L \to \mathbb{R}$ s.t.
  - $df = (e^t \lambda_0)|_L$;
  - $f|_{(-\infty, -T_-) \times \Lambda_-}$ and $f|_{(T_+, \infty) \times \Lambda_+}$ are constant.



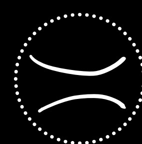
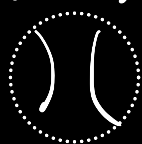
Cobordism = critical point degeneration $\rightsquigarrow$ relationship btwn invariants

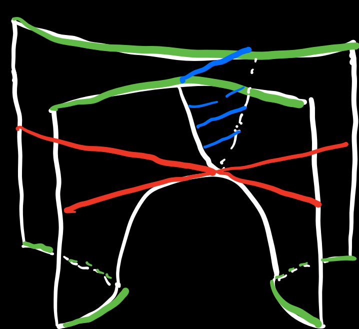
exact Lagrangian cobordisms $\rightsquigarrow$ DGA maps

Note: The problem of finding a stable tame iso. for

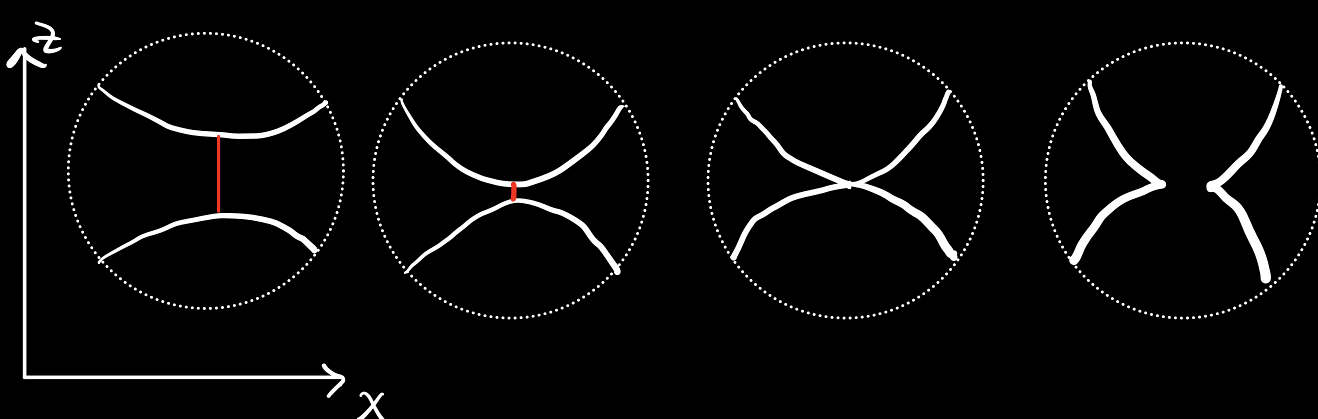


is the problem of finding the DGA map for the corresponding cobordism.

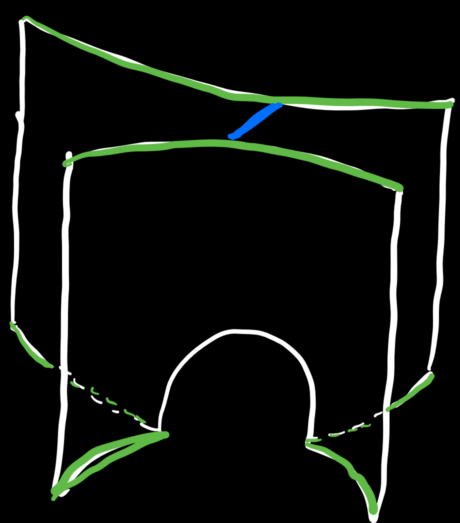
Topologically: a saddle can turn  into 



Can we do an analogous thing in a front projection?



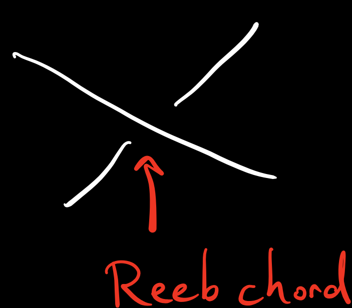
Claim: We can make this saddle exact Lagrangian. i.e., this cobordism does exist in our more restrictive geometric setting



← This picture is in $\mathbb{R}^3_{t,x,z}$ (front drops y)

The chord that we collapsed was a Reeb chord — entirely in z-direction.

In Lagrangian projection, these are double points:



Some Reeb chords are not contractible

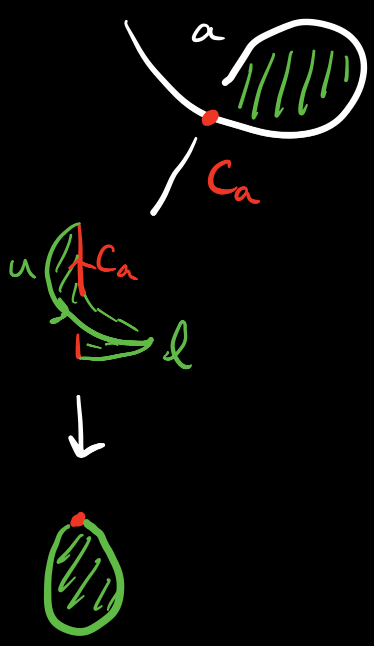
Stokes' Theorem

$\dim M = k+1$

$\deg \eta = k$

$$\int_{\partial M} \eta = \int_M d\eta$$

Consider $u: D^2 \to \mathbb{R}^3_{x,y,z}$ projecting to the green disc shown.



area enclosed by u in xy-plane

$$= \int_u dx \wedge dy = \int_u d(dz - y\,dx)$$

Stokes'

$$= \int_{\partial u} dz - y\,dx = \underbrace{\int_{\ell} (dz - y\,dx)}_{0,\ b/c\ \ell \subset \Lambda} + \int_{c_a} (dz - y\,dx)$$

$(dz - y\,dx)|_{\Lambda} \equiv 0$

c_a is only in z-dir.

$$= \int_{c_a} (dz - y\,dx) = \int_{c_a} dz = \text{length}(c_a)$$

So length(c_a) = area enclosed by loop.

No planar isotopy of the Lagr. projection which will collapse the area of this loop to be arbitrarily small.

If we can collapse our Reeb chord, then the picture in Lagrangian proj is