DGA :

$A = \bigoplus_{n=-\infty}^{\infty} A_n$

each A_n is R-module s.t. $A_m A_n \subseteq A_{m+n}$

R commutative, unital ring.

A : differential graded algebra over R.

(Co)-chain complex structure whose ∂ maps raise / lower degree by 1, and satisfies:

i) $\partial^2 = 0$

ii) $\partial(x \cdot y) = (\partial x) \cdot y + (-1)^{\deg(x)} x (\partial y)$

$\forall$ homogeneous elements $x, y \in A$.

- $\cdots \to C_{-1} \xrightarrow{\partial_{-1}} C_0 \xrightarrow{\partial_0} C_1 \xrightarrow{\partial_1} C_2 \xrightarrow{\partial_2} \cdots \to C_n \xrightarrow{\partial_n} C_{n+1} \to \cdots$

$\partial^2 = 0 \quad \Rightarrow \quad \partial_n \circ \partial_{n+1} = 0$

$\Rightarrow \quad Im(\partial_n) \subseteq Ker(\partial_{n+1})$.

- Defn : The sequence is "exact" if $Im(\partial_n) = Ker(\partial_{n+1})$ $\forall n$.
- If $x \in C_n$ is such that $\partial_n(x) = 0$, then x is a (co)-cycle.
- If $x \in C_n$ is in image of ∂_{n-1} (∂_{n+1}), then x is a (co)boundary.
- Cohomology groups :

$$CH_n(A) := Ker(\partial_{n+1}) \big/ Im(\partial_n)$$

- Cohomology of cochain complex : $\bigoplus_n CH_n(A)$

- <u>Chain maps</u> : Given Chain complex (A_1, d_1) and (A_2, d_2), a chain map $(A, d_A) \rightarrow (B, d_B)$ is a sequence of homomorphisms $f_n : A_n \rightarrow B_n \;\forall\; n$ that commutes with ∂-operators on the two chain complexes, i.e.

$$d_{B,n} \circ f_n = f_{n+1} \circ d_{A,n}$$

<u>Note</u> :- If $d_{A,n}(x) = 0$, then $d_{B,n}(f_n(x)) = f_{n+1}(d_{A,n}(x)) = 0$.

- If $x = d_{A,n-1}(y)$, then $d_{B,n-1}(f_{n-1}(y)) = f_n(\underbrace{d_{A,n-1}(y)}_{x})$

Thus chain maps take (co)cycles to (co) cycles and (co) boundaries to (co) boundaries.

$\Rightarrow$ induces a map on homology of chain sequence.

- <u>Example</u> : Given $R, R' \underset{op}{\subset} \mathbb{R}^3$, and $f : R \rightarrow R'$.

Recall $\Omega^{\circ}(R) = \bigoplus_{i=0}^{3} \Omega^i(R)$

Given $g \in \Omega^0(R')$, $f \circ g \in \Omega^0(R)$.

In general, given $\eta \in \Omega^i(R')$, $f^*\eta \in \Omega^i(R)$.
$\hookrightarrow$ pullback form.

Thus f defines a chain map between $\Omega^{\circ}(R)$ and $\Omega^{\circ}(R')$.

In general, the homology of this chain complex records topological information about R.

e.g. if R contains an i-dimensional hole then an i-form will be closed but not exact.

This is called De-Rham cohomology.

- <u>Computational tricks</u>:

  - If modules are infinitely-gen (as modules), cohomology might be infinitely generated.

  - Algebra structure causes problems: even if we're finitely gen. as algebra, can be infinitely generated as a module.

  - hard to look at two DGAs' homologies and determine if they are the same.

- <u>Example</u>: DGA for Chekanov knot I (e.g. 4.12 in Etnyre's notes)

$a_i, i = 1, \ldots 9$. Their gradings are

$$|a_i| = 1, \quad i = 1 \ldots 4,$$
$$|a_5| = 2,$$
$$|a_6| = -2,$$
$$|a_i| = 0, \quad i = 7 \ldots 9$$

and the boundary map is

$$\partial a_1 = 1 + a_7 + a_7 a_6 a_5,$$
$$\partial a_2 = 1 + a_9 + a_5 a_6 a_9,$$
$$\partial a_3 = 1 + a_8 a_7,$$
$$\partial a_4 = 1 + a_9 a_8,$$
$$\partial a_i = 0 \quad i \geq 5.$$

- Call a DGA augmented if ∂a does not contain any constant terms $\forall\, a \neq 0$. (above is not augmented).

- For an augmented DGA (A, d), consider

  $A_1 :=$ sub-algebra consisting of all words of length 1.

  $d_1 :=$ differential d but picks out only words of length 1

  e.g. if $d\alpha = \alpha_1 + \alpha_2\alpha_3 + \beta_1\beta_2\beta_3$,

  $d_1\alpha = \alpha_1$.

Homology of $(A_1, d_1) =:$ Linearised homology of (A, d).

- Augmentation : (A, ∂) DGA over $\mathbb{Z}_2$.

$$\epsilon : \mathcal{A} \to \mathbb{Z}_2$$

is called an *augmentation* of $\mathcal{A}$ if $\epsilon(1) = 1$, $\epsilon \circ \partial = 0$ and ϵ vanishes on any element of non-zero degree.

- Fact : Given (A, ∂) and augmentation ε, define

$$g_\varepsilon : A \to A, \quad \alpha \mapsto \alpha + \varepsilon(\alpha).$$

$$\partial^\varepsilon : g_\varepsilon \partial g_\varepsilon^{-1}.$$

Then $(A, \partial^\varepsilon)$ is an augmented DGA.

- <u>Back to Example</u>

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$$\begin{aligned} |a_i| &= 1, \quad i = 1 \ldots 4, \\ |a_5| &= 2, \\ |a_6| &= -2, \\ |a_i| &= 0, \quad i = 7 \ldots 9 \end{aligned}$$

and the boundary map is

$$\begin{aligned} \partial a_1 &= 1 + a_7 + a_7 a_6 a_5, \\ \partial a_2 &= 1 + a_9 + a_5 a_6 a_9, \\ \partial a_3 &= 1 + a_8 a_7, \\ \partial a_4 &= 1 + a_9 a_8, \\ \partial a_i &= 0 \quad i \geq 5. \end{aligned}$$

(worked out as per Etnyre's notes).