Recall: $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is only invariant up to stable tame isomorphism.

Hard to know when two DGAs are stable tame isomorphic.

We know that cohomology does not change under stable tame isomorphism, so we could compute

$$CH_*(\Lambda) = \frac{\ker \partial}{\operatorname{im} \partial}$$

and use this to distinguish $(\mathcal{A}_\Lambda, \partial_\Lambda)$ from $(\mathcal{A}_{\Lambda'}, \partial_{\Lambda'})$ (and thus Λ from Λ').

But this is still hard, as our algebra is non-commutative, and we're usually just working with a presentation of the homology.

Ex $CH_*(\text{trefoil}) = \langle a_3, a_4, a_5 \mid a_3 a_4 a_5 = 0 = a_5 a_4 a_3 \rangle$

One solution is to "approximate" $CH_*(L)$.

For the rest of today, let's turn $(\mathcal{A}_\Lambda, \partial_\Lambda)$ into a DGA over $\mathbb{Z}/2\mathbb{Z}$ by setting $t=1$ and reducing mod 2. (Probably also assuming $\mathrm{rot}(\Lambda) = 0$.)

So $\mathcal{A}_\Lambda = \mathbb{Z}/2\mathbb{Z}\langle a_1, \dots, a_n \rangle$.

Def'n. Denote by $\mathcal{A}_k$ the module generated by words in $a_1, \dots, a_n$ of length at most k.

$\mathcal{A}_k$ is NOT an algebra

Def'n. Call a DGA $(\mathcal{A}, \partial)$ augmented if the constant term of ∂a is 0, for every generator $a \in \mathcal{A}$.

If $(\mathcal{A}, \partial)$ is augmented, we can define

$$\partial_k : \mathcal{A}_k \to \mathcal{A}_k$$

by $\partial_k(w) := (\pi_k \circ \partial)(w)$, where

$$\pi_k : \mathcal{A} \to \mathcal{A}_k$$

throws out words of length $> k$.

Since ∂ decreases grading by 1, so does ∂_k.
Given a (module) generator $w \in \mathcal{A}_k$, we can write $\partial w = \partial_k w + u_k$, where u_k is a formal sum of words of length $> k$. Then

$$0 = \partial^2 w = \partial(\partial_k w) + \partial u_k$$
$$= \partial_k^2 w + v_k + \partial u_k,$$

where v_k is the "long" part of $\partial(\partial_k w)$.
Projecting to $\mathcal{A}_k$ gives

$$0 = \partial_k^2 w + \partial u_k.$$

The fact that $(\mathcal{A}, \partial)$ is augmented tells us that $\pi_k(\partial u_k) = 0$, so $0 = \partial_k^2 w$.

<u>Upshot</u>. $(\mathcal{A}_k, \partial_k)$ is a chain complex.

<u>Remarks</u> ① $(\mathcal{A}_k, \partial_k)$ is not a DGA
(can't multiply)

② The notation isn't great — $\mathcal{A}_k$ is a graded module, but k isn't related to the grading.

Def. If $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is augmented, we call the homology of $(\mathcal{A}_k, \partial_k)$ the order k contact homology of $(\mathcal{A}_\Lambda, \partial_\Lambda)$, and write

$$L_k CH_*(\mathcal{A}_\Lambda, \partial_\Lambda).$$

We call $L_1 CH_*(\mathcal{A}_\Lambda, \partial_\Lambda)$ the linearized contact homology. The order k Chekanov-Poincaré polynomial of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is

$$P_k(\lambda) := \sum_{i \in \mathbb{Z}} \dim(L_k CH_i(\mathcal{A}_\Lambda, \partial_\Lambda))\, \lambda^i.$$

Problems:
- Plenty of DGAs are not augmented.
- Stable tame isomorphisms can change $P_k(\lambda)$, so this is not a Leg. isotopy invariant.

So we've produced something which is computable, but it's not invariant. With a bit of care, we can fix this.

Assuming we still have $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}\langle a_1, \dots, a_n \rangle$, set

$$G(\mathcal{A}) := \left\{ \begin{array}{c} \text{graded automorphisms} \\ g : \mathcal{A} \to \mathcal{A} \end{array} \middle| \begin{array}{c} g(a_i) = a_i + c_i, \\ 1 \le i \le n, \quad c_i \in \mathbb{Z}/2\mathbb{Z} \end{array} \right\}.$$

Given $g \in G(\mathcal{A})$, we can define

$$\partial^g : \mathcal{A} \to \mathcal{A}$$
$$a \mapsto (g \circ \partial \circ g^{-1})(a).$$

<u>Exercise.</u> Check that ∂^g is a differential.

Then $(\mathcal{A}, \partial^g)$ is a DGA, and $g : \mathcal{A} \to \mathcal{A}$ is a tame isomorphism from $(\mathcal{A}, \partial)$ to $(\mathcal{A}, \partial^g)$.

$$\left(\partial^g \circ g = (g \circ \partial \circ g^{-1}) \circ g = g \circ \partial \right).$$

We can build $(\mathcal{A}, \partial^g)$ for every $g \in G(\mathcal{A})$, and maybe some of these will be augmented.

Define

$$G_a(\mathcal{A}) := \{ g \in G(\mathcal{A}) \mid (\mathcal{A}, \partial^g) \text{ is aug.} \}.$$

augmented.

Thm (Chekanov, 2002) The sets

$$I_k(\Lambda) := \{P_k(\lambda) \text{ for } (\mathcal{A}_\Lambda, \partial^g_\Lambda) \mid g \in G_a(\mathcal{A}_\Lambda)\}$$

and

$$L_k CH_*(\Lambda) := \{L_k CH_*(\mathcal{A}_\Lambda, \partial^g_\Lambda) \mid g \in G_a(\mathcal{A}_\Lambda)\}$$

are Legendrian isotopy invariants.

Okay, so now we have something which is invariant under Legendrian isotopy, and is *relatively* computable.

We can make $G_a(\mathcal{A}_\Lambda)$ even more computable via augmentations.

Recall that an augmentation (in our current setting) is a DGA map

$$\varepsilon : \mathcal{A}_\Lambda \to \mathbb{Z}/2\mathbb{Z}$$

s.t.
- $\varepsilon(1) = 1$;
- $\varepsilon \circ \partial = 0$;
- $\varepsilon(a) = 0$, if $|a| \neq 0$.

Lemma. $G_a(\mathcal{A}_\Lambda) = \{a_i \mapsto a_i + \varepsilon(a_i) \mid \varepsilon \text{ is an aug.}\}$.

So we have a strategy for distinguishing Λ & Λ'!

1. Compute the Chekanov-Eliashberg DGAs $(\mathcal{A}_\Lambda, \partial_\Lambda)$ & $(\mathcal{A}_{\Lambda'}, \partial_{\Lambda'})$.
2. Find all augmentations of both DGAs.
3. Use these to compute the sets $I_k(\Lambda)$, $L_k CH_*(\Lambda)$, $I_k(\Lambda')$, & $L_k CH_*(\Lambda')$.
4. Look for differences.

Ex For the Chekanov pair Λ_1, Λ_2, we've found

$$\mathcal{A}_{\Lambda_1} = \mathbb{F}_2\langle a_1, \dots, a_9\rangle \qquad \mathcal{A}_{\Lambda_2} = \mathbb{F}_2\langle b_1, \dots, b_9\rangle$$

$$|a_i| = \begin{cases} 1, & 1 \le i \le 4 \\ 0, & 5 \le i \le 9 \end{cases} \qquad |b_i| = \begin{cases} 1, & 1 \le i \le 4 \\ 0, & 7 \le i \le 9 \end{cases}$$

$$|b_5| = 2, \quad |b_6| = -2.$$

$$\partial a_1 = 1 + a_7 + a_7 a_6 a_5 + a_5 + a_9 a_8 a_5 \qquad \partial b_1 = 1 + b_7 + b_7 b_6 b_5$$

$$\partial a_2 = 1 + a_9 + a_5 a_6 a_9 \qquad \partial b_2 = 1 + b_9 + b_5 b_6 b_9$$

$$\partial a_3 = 1 + a_8 a_7 \qquad \partial b_3 = 1 + b_8 b_7$$

$$\partial a_4 = 1 + a_9 a_8 \qquad \partial b_4 = 1 + b_9 b_8$$

$|G_a(A_{\Lambda_1})| = 3$

$\varepsilon_1(a_5) = 0,\ \varepsilon_1(a_6) = 0$

$\varepsilon_2(a_5) = 0,\ \varepsilon_2(a_6) = 1$

$\varepsilon_3(a_5) = 1,\ \varepsilon_3(a_6) = 0$

$\varepsilon_i(a_7) = \varepsilon_i(a_8) = \varepsilon_i(a_9) = 1$

$|G_a(A_{\Lambda_2})| = 1$

$$\varepsilon'(b_i) = \begin{cases} 0, & i \le 6 \\ 1, & i \ge 7 \end{cases}.$$

At this point, we still can't tell the knots apart.

But we can compute $L_1CH_*(\Lambda_i),\ i = 1, 2.$

For Λ_1:

- using ε_1. $\partial^{\varepsilon_1} = g_{\varepsilon_1} \circ \partial \circ g_{\varepsilon_1}^{-1}$

  So $\partial^{\varepsilon_1}(a_i) = (g_{\varepsilon_1} \circ \partial)(a_i + 1)$

  $= g_{\varepsilon_1}(\partial a_i) = g_{\varepsilon_1}(0) = 0,\quad 7 \le i \le 9.$

  For $1 \le i \le 6$, $\partial^{\varepsilon_1}(a_i) = (g_{\varepsilon_1} \circ \partial)(a_i)$

  $$\begin{aligned} \partial^{\varepsilon_1}(a_1) &= g_{\varepsilon_1}(1 + a_7 + a_7a_6a_5 + a_5 + a_9a_8a_5) \\ &= 1 + (1+a_7) + (1+a_7)a_6a_5 + a_5 \\ &\quad + (1+a_9)(1+a_8)a_5 \\ &= a_7 + a_6a_5 + a_7a_6a_5 + a_9a_5 + a_8a_5 + a_9a_8a_5 \end{aligned}$$

  $\therefore\ \partial_1^{\varepsilon_1}(a_1) = a_7$

$$\partial^{\varepsilon_1}(a_2) = g_{\varepsilon_1}(1 + a_9 + a_5 a_6 a_9)$$

$$= 1 + (1 + a_9) + a_5 \cdot a_6 \cdot (1 + a_9)$$

$$= a_9 + a_5 a_6 + a_5 a_6 a_9$$

$$\therefore \ \partial_1^{\varepsilon_1}(a_2) = a_9$$

$$\partial^{\varepsilon_1}(a_3) = g_{\varepsilon_1}(1 + a_8 a_7)$$

$$= 1 + (1 + a_8)(1 + a_7)$$

$$= a_8 + a_7 + a_8 a_7$$

$$\therefore \ \partial_1^{\varepsilon_1}(a_3) = a_8 + a_7$$

$$\partial^{\varepsilon_1}(a_4) = g_{\varepsilon_1}(1 + a_9 a_8)$$

$$= 1 + (1 + a_9)(1 + a_8)$$

$$= a_9 + a_8 + a_9 a_8$$

$$\therefore \ \partial_1^{\varepsilon_1}(a_4) = a_9 + a_8$$

$$\therefore \ [\partial_1^{\varepsilon_1}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{Ker} \partial_1^{\varepsilon_1} = \mathbb{F}_2 \langle a_5, a_6, a_7, a_8, a_9, a_1 + a_2 + a_3 + a_4 \rangle$$

$$\operatorname{im} \partial_1^{\varepsilon_1} = \mathbb{F}_2 \langle a_7, a_8, a_9 \rangle$$

$\therefore\ L_1 CH_*(\mathcal{A}_{\Lambda_1}, \partial^{\varepsilon_1}) = \mathbb{F}_2 \langle \underbrace{a_1 + a_2 + a_3 + a_4}_{\text{grading } 1}, \underbrace{a_5, a_6}_{\text{grading } 0} \rangle$

$\therefore\ P_1(\lambda) = 2 + \lambda$

Check: Get the same $P_1(\lambda)$ for ε_2 and ε_3.

But yesterday we saw that $P_1(\lambda) = \lambda^{-2} + \lambda + \lambda^2$ for Λ_2 with its unique augmentation.

$\therefore$ distinct knots!