Product Structure on Pairs in Legendrian Knot Atlas

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Abstract

Linearized contact (co)homology, an invariant calculated from the Chekanov-Eliashberg DGA of a Legendrian knot, can distinguish knots with the same classical invariants. A recently developed invariant, product structure—composing of cup, Massey, and higher-order Massey product structures, preserves some lost information in the linearization process. In this report, we apply these product structure invariants to investigate a pair of $m(7_7)$ Legendrian knots which are believed to be distinct given in Legendrian knot atlas, and we present the detailed computation of obtaining their isomorphic cup product structure. We prove a proposition which characterizes a sufficient condition for Massey products or higher-order Massey products to be trivial. We then apply this proposition to our pair of knots, determining that product structure is insufficient to distinguish the pair, and we conclude that product structure cannot distinguish any pair in the atlas.

1 Introduction

Classical invariants for Legendrian knots introduced in [4], such as Thurston-Bennequin number and rotation number, have been used to compare two Legendrian knots. Two Legendrian knots are not Legendrian isotopic (i.e., distinct) if they have different invariant quantities, though having the same invariants does not necessarily imply that the Legendrian knots are Legendrian isotopic.

By the results shown in [5], cup product, Massey product, and higher-order Massey product structures are Legendrian isotopy invariants. We use these invariants to compare two $m(7_7)$ knots in the atlas given in [2] in hope that these two knots can be distinguished.

From the atlas, we chose two $m(7_7)$ knots represented in Figure 1.0.1. They are better candidates than any other pair that is believed to be distinct in the atlas for obtaining product structure, because they have augmentations and we could apply product structure to potentially distinguish them. The reasons we chose knots with augmentations are:

- (Higher-order) Massey product is defined using an A_∞ structure on linearized contact cohomology, which requires augmented differentials; thus without augmentations, we would not be able to obtain product structure.

- In the atlas of [2], the Poincaré-Chekanov polynomial of the two knots is given, so we use it as a reality check, making sure we get the correct DGAs.

Though the product structure is shown to not be able to distinguish this pair, we proved a proposition that characterizes some criteria to determine if the product structure is trivial. As a result, no pair in the atlas can be distinguished by their product structure.

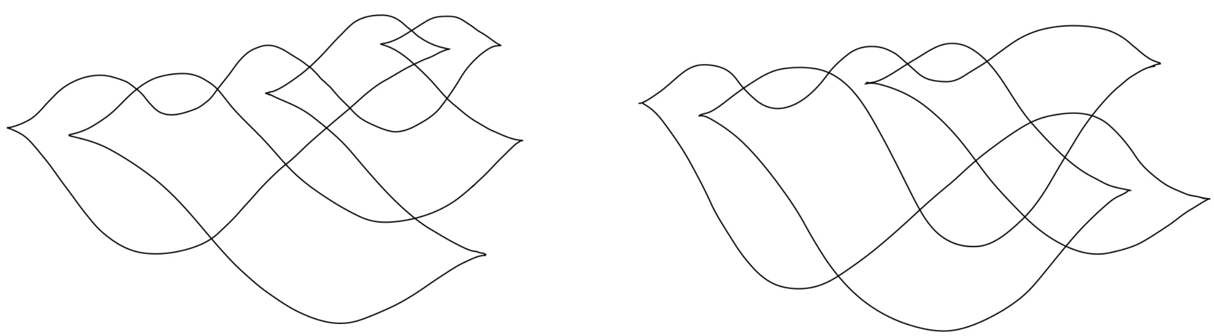

Figure 1.0.1: Front projections of $m(7_7)$ knots (Λ_1 on the left, Λ_2 on the right)

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2 Background

2.1 Legendrian Knot and Front Projection

In this report, the knots we are concerned with are Legendrian knots in the standard contact 3-manifold, $(\mathbb{R}^3, \xi_{std})$, where $\xi_{std} = \ker \alpha$, and α is the standard contact form $dz - ydx$.

Definition. *A knot Λ is **Legendrian** if it has a regular parametrization $\varphi\colon \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ defined by $\theta \longmapsto (x(\theta), y(\theta), z(\theta))$ such that for all θ,*

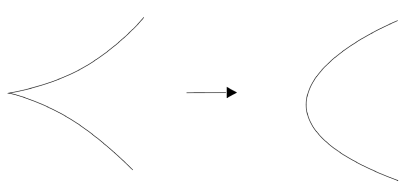
$$z'(\theta) - y(\theta)x'(\theta) = 0,$$

that is,

$$\alpha_{\varphi(\theta)}(\varphi'(\theta)) = 0.$$

Several types of diagrams are used to represent a Legendrian knot in this report: front projection, Lagrangian projection, and grid diagram.

Front projection is the projection of a Legendrian knot into the xz-plane. By the condition imposed on the knot, $z'(\theta) - y(\theta)x'(\theta) = 0$, we cannot have vertical tangency in the front projection. **Lagrangian projection** is the projection of a Legendrian knot into the xy-plane. The geometric conditions placed on Legendrian knots allow us to consider a well-defined *resolution* which converts a front projection into a Lagrangian projection, and vice versa. To convert a front projection into a Lagrangian projection, left cusps are smoothed out and right cusps are smoothed and twisted to add an extra crossing in the resulting projection, as shown in Figure 2.1.1. **Grid diagram** is a computer-recognizable version of front projection, and its details are left to the next subsection.


(1)

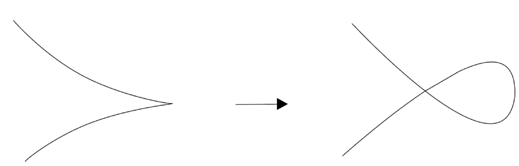

(2)

Figure 2.1.1: (1) Resolution of a left cusp; (2) Resolution of a right cusp

Similar to the effect that Reidemeister moves have on topological knots, Legendrian Reidemeister moves (shown in Figure 2.1.2), when applied to the front projection of a Legendrian knot, does not change its Legendrian isotopy type. The following theorem in [6] further characterizes the relation between Legendrian Reidemeister moves and Legendrian isotopy.

Theorem 2.1.1. *Two front projections represent Legendrian isotopic Legendrian knots if and only if they are related by a finite sequences of Legendrian Reidemeister moves and regular homotopies.*

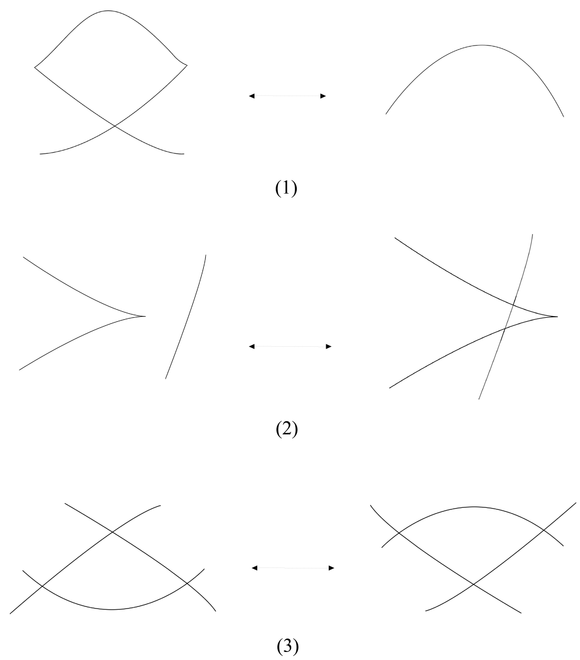


Figure 2.1.2: Three Legendrian Reidemeister moves

2.2 Grid Diagram and Cromwell Moves

As introduced in [2], a **grid diagram** is an $n \times n$ grid with each of n X's and n O's occupying one square distinct from another such that only one X and one O are in each row and in each column. By connecting O's to X's horizontally and X's to O's vertically, and let the vertical strand come in front of the horizontal one whenever there is a crossing, we obtain its associated oriented link or knot. To convert the grid diagram into front projection, we rotate the grid diagram 45° counterclockwise and smooth corners pointing towards horizontal direction.

Given a grid diagram, we can perform **Cromwell moves** on it: they are modifications to the grid without changing the smooth isotopy type of the Legendrian knot the grid diagram represents. A subset of Cromwell moves, which plays a similar role as Reidemeister moves for the front projection, does not change the Legendrian isotopy type of the associated Legendrian knot of a grid diagram. This subset contains

- **torus translation**: move the leftmost (or rightmost) column of the grid to rightmost (or leftmost) column, or move the uppermost (bottommost) row to bottommost (or uppermost) row;

- **commutation**: switch two adjacent rows (or columns) in which segments connecting O's to X's (or X's to O's) are either disjoint or nested when projected to a single horizontal (or vertical) line;

- **X:NE** and **X:SW** stabilization and destabilization: two of four possible X stabilizations displayed in 2.2.1, which consist of adding a row and a column to the grid in a way that preserves the Legendrian isotopy class.

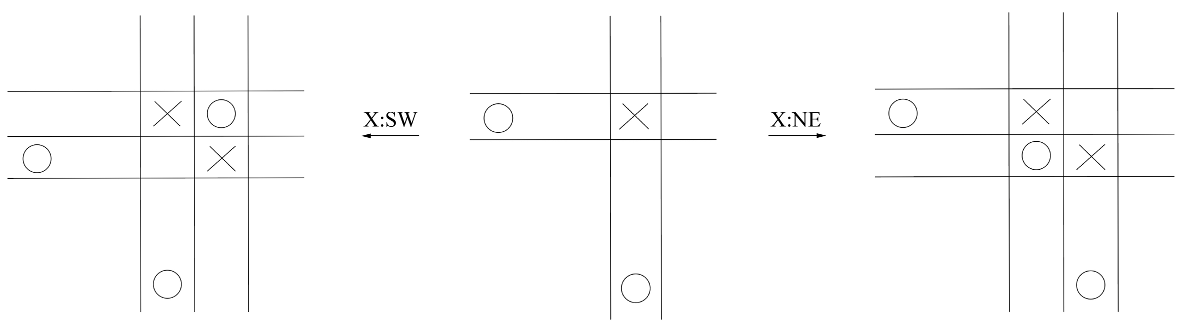


Figure 2.2.1: Two types of Legendrian-isotopy-preserving X stabilizations

2.3 Chekanov-Eliashberg DGA

Ckekanov, following ealier work of Eliashberg, constructed the Chekanov-Eliashberg DGA, a means of assigning to any given Legendrian knot a differential graded algebra that can be used to calculate newer invariants [1, 3]. There are many different ways to define this DGA, preserving various levels of information, and having various degrees of computability. Our project focuses on Legendrian knots which have rotation number 0, which allows us to use a simplified version of the Chekanov-Eliashberg DGA, as will be explained during this section. This subset of Legendrian knots will be useful because having rotation number 0 is necessary (although not sufficient) for the existence of augmentations to the base field, which will be necessary for further calculations.

The algebra that we will use is defined over $\mathbb{Z}_2$ and generated by the Reeb chords of our knot. In the Lagrangian projection these correspond to the crossings. The geometric intuition for the DGA comes from considering a knot's Lagrangian projection, but it can be calculated combinatorially from the front projection. When defining the DGA from the front projection (as we have done in our research) the generators correspond to crossings and right cusps, since right cusps become crossings when converting from front to Lagrangian projections.

It is then simple to assign a grading to each of these generators. Generators associated with a right cusp have a grading of 1, and for generators associated with a crossing we must consider a *capping path*. This is simply a path traced out on the front projection of our knot from the overstrand of the crossing to the understrand. For each capping path γ we will denote $D(\gamma)$ as the number of cusps travelled downwards, and $U(\gamma)$ as the number of cusps travelled upwards. Then the generator associated with a crossing has a grading of $D(\gamma) - U(\gamma)$. Of course, for each crossing there will be two distinct capping paths, but for knots with rotation number 0, both paths will give the same grading for each generator. Defining the grading of generators on other Legendrian knots requires more detail, which we will not need to consider for this report.

The differential is the most complicated part of the DGA to define. For each generator a_i we must consider maps of the unit disk with some number of boundary punctures to the xz-plane, such that the maps are immersions of the disk into the knot that meet some specific criteria:

1. Each of the boundary punctures of the disk must be mapped to a crossing or right cup of the projection;

2. The first boundary puncture must mapped to a_i such that the immersion covers a left or right quadrant of the associated crossing;

3. All other boundary punctures must be mapped to generators in such a way that the immersion covers an upper or lower quadrant of the associated crossings.

Then we assign a word to each immersion, which is a noncommutative product of all the generators mapped to along the boundary punctures of the disk (except a_i). Then the differential of a_i is the formal sum of the words of all possible immersions associated with that generator (with an extra $+1$ if the generator a_i denotes a right cusp). More information on this DGA, including examples, can be found in [4].

2.4 Linearized Contact (Co)homology

Defined on a (co)chain complex—in our case, Chekanov-Eliashberg DGA as above—the (co)homology measures the failure of exactness. The (co)homology of the C.E. DGA is called **Legendrian contact homology** (LCH), a Legendrian isotopy invariant.

Since the computability and usefulness of LCH are often in contrary positions, to strike a balance between the two, we often use augmentation and linearization to obtain a modified differential, thus resulting in **linearized contact (co)homology**.

Definition. *Given $(\mathcal{A}, \partial)$ a DGA, a map $\varepsilon : \mathcal{A} \longrightarrow \mathbb{Z}/2$ is an **augmentation** if*

- $\varepsilon(1) = 1$,
- $\varepsilon \circ \partial = 0$, *and*
- $\varepsilon(x) = 0$ *whenever* $|x| \neq 0$.

Note that an augmentaion does not always exist for a DGA, and a DGA might admit several augmentaions. When at least one augmentation exists, we obtain a linearized differential by first considering a map $g_\varepsilon : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$g_\varepsilon : a \longmapsto a + \varepsilon(a),$$

thus resulting in **augmented differential**

$$\partial^\varepsilon = g_\varepsilon \partial g_\varepsilon^{-1},$$

then for each augmentation ε of $(\mathcal{A}, \partial)$, after eliminating nonlinear terms of ∂^ε, we have a linearized chain complex $(\mathcal{A}, \partial^\varepsilon)$. By taking $\mathcal{A}'$ with basis dual to basis of $\mathcal{A}$ and δ^ε the adjoint of ∂^ε, we have a cochain complex $(\mathcal{A}', \delta^\varepsilon)$, and homologies of these two complexes are **linearized contact homology** and **linearized contact cohomology**, denoted by LCH_*^ε and LCH_ε^*, respectively.

2.5 A_∞-Structure

Our research focus being knots with identical linearized Legendrian contact homology, we needed to study and utilize techniques for extracting information lost during the process of linearization. Our major tools to do this were the product structures defined in [5] by Civan, Entyre, Koprowski, Sabloff, and Walker. This strategy begins with an augmented differential and uses it to define an A_∞-structure. This involves defining an infinite collection of maps

m_n, $n \in \mathbb{Z}^+$, where the map m_n inputs a permutation of n generators from the DGA and outputs information about where this permutation appears in the augmented differential. The maps m_n must satisfy the A_∞ relations. The relations for m_1 is $m_1 \circ m_1 = 0$, and for m_2 is $m_1(m_2(a, b)) = m_2(m_1(a), b) + (-1)^{|a|} m_2(a, m_1(b))$. The rest can be found in [5].

More precisely, if the word $a_1 ... a_k$ appears in the differentials ∂a_1,...,∂a_i then we have:

$$m_k(a_1, ..., a_k) = a_1 + ... + a_i.$$

If the word $a_1, ..., a_k$ is not a term in the augmented differential then we have $m_k(a_1, ..., a_k) = 0$. Of course, because the augmented differential contains only finitely many terms, only a relatively small amount of terms $m_k(a_1, ..., a_k)$ will be non-trivially defined. However, this A_∞-structure is a method of expressing the nonlinear information from a knot's augmented differential in a way that can be used in calculations, particularly in the calculations of cup products, Massey products, and higher-order Massey producs, which is the next idea discussed by Civan, Entyre, Koprowski, Sabloff, and Walker. It is also worth noting that terms such as $m_2(a_1 + a_2, a_3)$ can be evaluated by distribution as $m_2(a_1, a_3) + m_2(a_2, a_3)$.

2.6 Cup Product Structure

Civan, Etynre, Koprowski, Sabloff, and Walker demonstrate how the A_∞-strucutre previously defined can be used to define a cup product structure for a knot. They begin by treating the m_1 map as a new differential and taking its homology. This can be done because the map is defined in such a way that it is itself linearized and one can check that $m_1 \circ m_1 = 0$. This will result in a collection of classes $LCH^n_\varepsilon \cong \mathbb{Z}_2\langle a^1_n, ..., a^{l_n}_n \rangle$ where each generator $a^1_n, ..., a^{l_n}_n$ corresponds to an equivalence class of elements in the kernel of m_1 with grading n.

The cup product structure is then defined on the finite set of all $a_k, ..., m_k$ where LCH^k_ε is nontrivial. It is defined as follows:

$$a_i \cup b_j = [m_2(a, b)] \text{ where } a \in a_i \text{ and } b \in b_j$$

We also have that the cup product $a_i \cup b_j$ will be an equivalence class with elements of grading $i + j + 1$. It will be important to remember that the cup product is distinct from the map m_2 because it is defined as an equivalence class of generators (and possibly sums of generators), not as an individual generator. This cup product structure is an invariant of Legendrian knots, so if two knots have cup product structures which are not isomorphic then they are distinct. For knots that have isomorphic cup product structures, (higher-order) Massey products can be used to attempt to distinguish them, as Civan, Entyre, Koprowski, Sabloff, and Walker next explain in the text.

2.7 (Higher-Order) Massey Product Structure

Cup product is generalized into Massey product, which is then further generalized into higher-order Massey product in [5]. Since the product structure is invariant under Legendrian isotopy, we can then use this structure to distinguish a pair of Legendrian knots if they satisfy certain conditions which we will introduce below.

Definition. *For $[a]$, $[b]$, $[c]$ elements in LCH^*_ε with gradings r, s, t, respectively, satisfying*

$$[a] \cup [b] = 0 = [b] \cup [c],$$

*then the **Massey product***

$$\{[a], [b], [c]\} \in \frac{LCH_\varepsilon^{r+s+t+1}}{([a] \cup LCH^*_\varepsilon + LCH^*_\varepsilon \cup [c]) \cap LCH_\varepsilon^{r+s+t+1}},$$

and it is given by the formula

$$\{[a], [b], [c]\} = [m_3(a, b, c) + m_2(a, x) + m_2(y, c)],$$

where x, y satisfy

$$m_1(x) = m_2(b, c),$$
$$m_1(y) = m_2(a, b).$$

Using A_∞-structure, higher-order Massey product is defined inductively in a similar way as the triple Massey product. More specifically, $\{[a_1], \ldots, [a_n]\}$ is an equivalence class of all ways of performing lower-order Massey products.

3 Work/Result

3.1 Perform Cromwell Moves

The original grid diagram of the $m(7_7)$ knots from [2] gives us front projections of the two knots in Figure 1.0.1. Notice that there is one right cusp being looped in another right cusp in both knots. This would make our future steps of finding the DGAs for the two knots difficult, since the original projections have immersed disks we would need to count. More details can be found in [4]. Therefore, before doing computations on the pair of $m(7_7)$ knot, we want to make them easier to deal with by performing the Cromwell moves on grid diagrams, as mentioned in previous section, so that the new projections allow us to count embedded disks only.

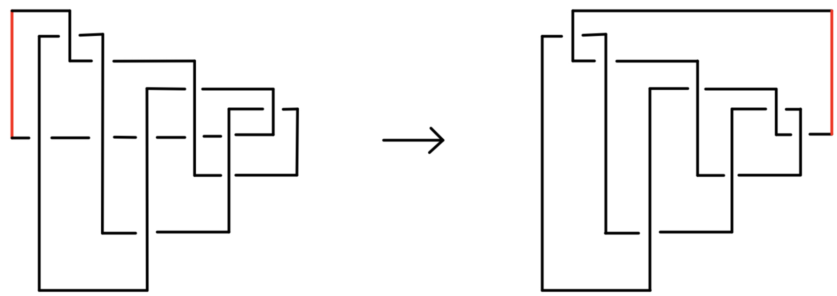


Figure 3.1.1: Λ_1, torus translation on grid diagram

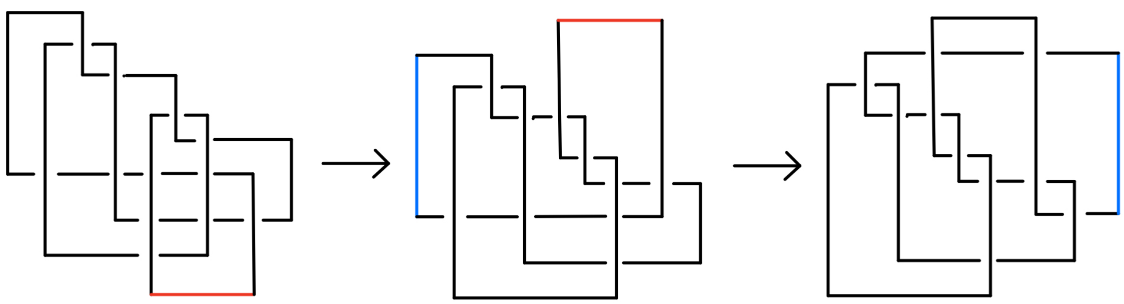


Figure 3.1.2: Λ_2, torus translations on grid diagram

For Λ_1, we performed one torus translation by moving the leftmost column to the rightmost column, as showing in Figure 3.1.1.

For Λ_2, we performed two torus translations by moving the bottommost row to the topmost row and moving the leftmost column to the rightmost column, as shown in Figure 3.1.2.

3.2 DGA Computation

We obtained the Chekanov-Eliashberg DGAs of the two $m(7_7)$ knots from their front projections—recall that they are converted from the grid diagrams by rotating them 45° counterclockwise and smoothing out the top and bottom corners.

The gradings for the right cusps are 1, and we get the gradings for the crossings by computing $D(\gamma) - U(\gamma)$ for a capping path γ.

The gradings for Λ_1 (labels of the generators are given in Figure 3.2.1):

$$1 = |a_1| = |a_2| = |a_3| = |a_4| = |b_3| = |b_7|,$$

$$0 = |b_1| = |b_2| = |b_5|,$$
$$-1 = |b_4| = |b_6|;$$

and the gradings for Λ_2 (labels of the generators are given in Figure 3.2.2):

$$1 = |a_1| = |a_2| = |a_3| = |b_5| = |b_8|,$$
$$0 = |b_1| = |b_2| = |b_3|,$$
$$-1 = |b_4| = |b_6| = |b_7| = |b_9|,$$
$$-2 = |b_{10}|.$$

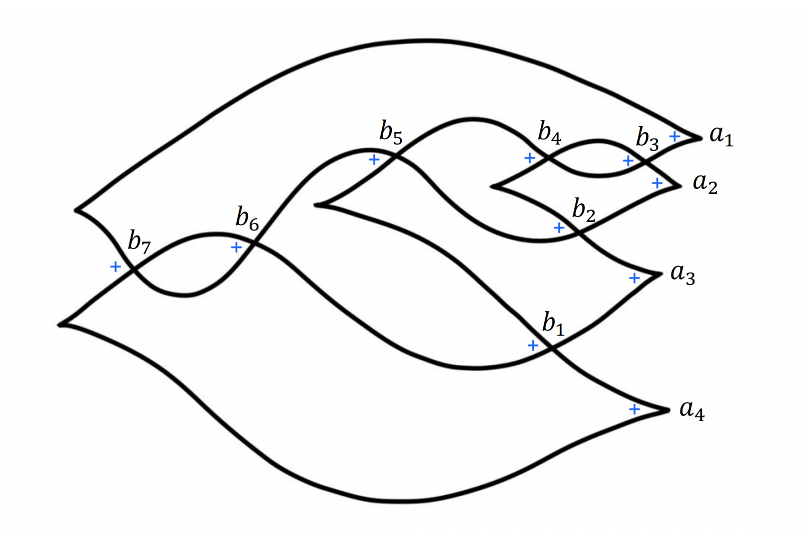


Figure 3.2.1: Front projection of Λ_1 after Cromwell move

We obtain the differentials by counting the disks starting and ending on each generator. When we start counter-clockwise at a generator and move along the strand, we can make turns at the crossings we pass. We can only make turns at up or down quadrants and cover at most two quadrants. Notice that each crossing makes four quadrants and each right cusp makes one—the reason is that each quadrant has positive or negative Reeb signs (in our figures the labeled " + " quadrants and their opposites are positive; the rest are negative,) and we can only turn at crossings which cover up a quadrant with negative Reeb sign. For example, for the first $m(7_7)$ knot, if we start from the right cusp a_1, we get four disks b_5, $b_5b_4b_3$, $b_7b_6b_5$, and $b_7b_6b_5b_4b_3$. Each b_i records each puncture at b_i crossing.

The differentials for Λ_1:

$$\partial a_1 = 1 + b_5 + b_5b_4b_3 + b_7b_6b_5 + b_7b_6b_5b_4b_3,$$

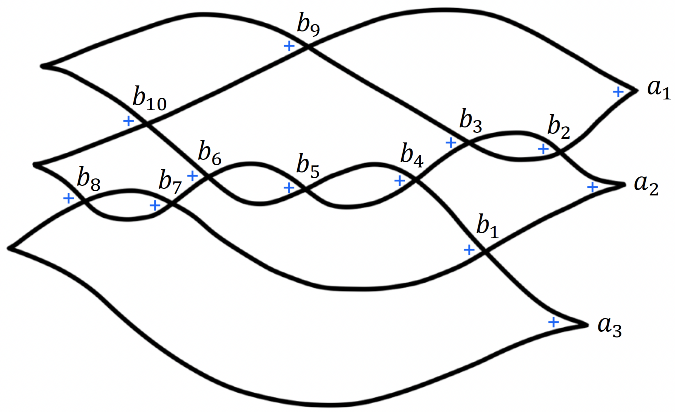


Figure 3.2.2: Front projection of Λ_2 after Cromwell moves

$$
\begin{aligned}
\partial a_2 &= 1 + b_2 + b_3 b_4 b_2,\\
\partial a_3 &= 1 + b_2 b_5 b_1,\\
\partial a_4 &= 1 + b_1 + b_1 b_6 b_7,\\
\partial b_i &= 0 \text{ for } i = 1, \dots, 7;
\end{aligned}
$$

for Λ_2:

$$
\begin{aligned}
\partial a_1 &= 1 + b_2 + b_5 b_4 b_2 + b_8 b_7 b_2 + b_8 b_7 b_5 b_4 b_2,\\
\partial a_2 &= 1 + b_2 b_1 + b_2 b_9 b_8 + b_5 b_{10} b_8 + b_2 b_9 b_5 b_1 + b_2 b_6 b_5 b_1 + b_2 b_3 b_5 b_{10} b_8 + b_2 b_9 b_8 b_7 b_5 b_1,\\
\partial a_3 &= 1 + b_1 + b_4 b_8 + b_6 b_8 + b_1 b_7 b_8 + b_4 b_5 b_6 b_8,\\
\partial b_1 &= b_{10} b_8 + b_4 b_5 b_{10} b_8,\\
\partial b_3 &= b_4 + b_6 + b_9 + b_6 b_5 b_4 + b_9 b_5 b_4 + b_9 b_8 b_7 + b_9 b_8 b_7 b_5 b_4,\\
\partial b_6 &= b_{10} + b_{10} b_8 b_7,\\
\partial b_9 &= b_{10},\\
\partial b_i &= 0 \text{ for } i = 2, 4, 5, 7, 8, 10.
\end{aligned}
$$

3.3 Augmented Differential Computation

By the conditions imposed on an augmentation as defined in previous section, we have the following results.

For Λ_1, there is only one augmentation ε such that $\varepsilon(b_1) = \varepsilon(b_2) = \varepsilon(b_3) = 1$ and the rest are zero. Thus, it has its augmented differentials:

$$\begin{aligned}
\partial^\varepsilon a_1 &= b_5 + b_4b_3 + b_7b_6 + b_5b_4b_3 + b_7b_6b_5 + b_7b_6b_4b_3 + b_7b_6b_5b_4b_3\\
\partial^\varepsilon a_2 &= b_2 + b_3b_4 + b_3b_4b_2\\
\partial^\varepsilon a_3 &= b_1 + b_2 + b_5 + b_2b_1 + b_2b_5 + b_5b_1 + b_2b_5b_1\\
\partial^\varepsilon a_4 &= b_1 + b_6b_7 + b_1b_6b_7\\
\partial^\varepsilon b_i &= 0 \text{ for } i = 1, \ldots, 7.
\end{aligned}$$

For Λ_2, there are two augmentations ε_1 and ε_2 such that $\varepsilon_1(b_1) = \varepsilon_1(b_2) = 1$, $\varepsilon(b_3) = 0$, and $\varepsilon_2(b_1) = \varepsilon_2(b_2) = \varepsilon_2(b_3) = 1$. Note that these two augmentations will result in the same augmented differentials, as b_3 does not appear in any term of any differential, so let ε invariably denote either augmentation. Thus, we have the following augmented differentials:

$$\begin{aligned}
\partial^\varepsilon a_1 &= b_2 + b_5b_4 + b_8b_7 + b_5b_4b_2 + b_8b_7b_2 + b_8b_7b_5b_4 + b_8b_7b_5b_4b_2\\
\partial^\varepsilon a_2 &= b_1 + b_2 + b_9b_8 + b_9b_5 + b_6b_5 + b_2b_1 + b_2b_9b_8 + b_2b_9b_5 + b_9b_5b_1 + b_2b_6b_5 + b_6b_5b_1 + b_5b_{10}b_8\\
&\quad + b_3b_5b_{10}b_8 + b_9b_8b_7b_5 + b_2b_9b_5b_1 + b_2b_6b_5b_1 + b_2b_3b_5b_{10}b_8 + b_2b_9b_8b_7b_5 + b_9b_8b_7b_5b_1\\
\partial^\varepsilon a_3 &= b_1 + b_7b_8 + b_4b_8 + b_6b_8 + b_1b_7b_8 + b_4b_5b_6b_8\\
\partial^\varepsilon b_1 &= b_{10}b_8 + b_4b_5b_{10}b_8\\
\partial^\varepsilon b_3 &= b_4 + b_6 + b_9 + b_6b_5b_4 + b_9b_8b_7 + b_9b_5b_4 + b_9b_8b_7b_5b_4\\
\partial^\varepsilon b_6 &= b_{10} + b_{10}b_8b_7\\
\partial^\varepsilon b_9 &= b_{10}\\
\partial^\varepsilon b_i &= 0 \text{ for } i = 2, 4, 5, 7, 8, 10;
\end{aligned}$$

3.4 Linearized Contact (Co)homology Calculation

After finding the augmented DGAs for our knots we were able to use these to confirm the linearized contact homology as given in the Legendrian knot atlas. While computing this invariant for both knots would not help us to tell them apart, it is a nice reality check on our differentials for the knots, thus a necessary and nontrivial task.

Given the augmented differentials, to calculate linearized contact homology, the first step is linearizing augmented differentials. Λ_1 has the following linearized differentials:

$$\begin{aligned}
\partial_1^\varepsilon a_1 &= b_5\\
\partial_1^\varepsilon a_2 &= b_2\\
\partial_1^\varepsilon a_3 &= b_1 + b_2 + b_5
\end{aligned}$$

$$\partial_1^\varepsilon a_4 = b_1$$
$$\partial_1^\varepsilon b_i = 0 \text{ for } i = 1, \ldots, 7.$$

Λ_2 has the following linearized differentials:

$$\begin{aligned}
\partial_1^\varepsilon a_1 &= b_2 \\
\partial_1^\varepsilon a_2 &= b_1 + b_2 + b_8 \\
\partial_1^\varepsilon a_3 &= b_1 \\
\partial_1^\varepsilon b_3 &= b_4 + b_6 + b_9 \\
\partial_1^\varepsilon b_6 &= b_{10} \\
\partial_1^\varepsilon b_9 &= b_{10} \\
\partial_1^\varepsilon b_i &= 0 \text{ for } i = 1, 2, 4, 5, 7, 8, 10.
\end{aligned}$$

To compute the LCH for our first knot, we consider the following maps:

$$\begin{aligned}
\partial_1 &: A_1 \to A_0 \\
\partial_0 &: A_0 \to A_{-1} \\
\partial_{-1} &: A_{-1} \to A_{-2}
\end{aligned}$$

These are used in the following calculations for homology:

$$\begin{aligned}
LCH_1 &= \ker(\partial_1) = \mathbb{Z}_2\langle a_1 + a_2 + a_3 + a_4, b_3, b_7\rangle \\
LCH_0 &= \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}_2\langle b_1, b_2, b_5\rangle}{\mathbb{Z}_2\langle b_1, b_2, b_5\rangle} = 0 \\
LCH_{-1} &= \ker(\partial_{-1}) = \mathbb{Z}_2\langle b_4, b_6\rangle
\end{aligned}$$

The Poincaré polynomial $3t + 2t^{-1}$ contains this information. Likewise, with similar maps ∂_1, ∂_0, ∂_{-1}, and ∂_{-2}, the following calculations for homology can be made for our second knot:

$$\begin{aligned}
LCH_1 &= \ker(\partial_1) = \mathbb{Z}_2\langle a_1 + a_2 + a_3, b_5, b_8\rangle \\
LCH_0 &= \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}_2\langle b_1, b_2\rangle}{\mathbb{Z}_2\langle b_1, b_2\rangle} = 0 \\
LCH_{-1} &= \frac{\ker(\partial_{-1})}{\operatorname{im}(\partial_0)} = \frac{\mathbb{Z}_2\langle b_4, b_7, b_6 + b_9\rangle}{\mathbb{Z}_2\langle b_4 + b_6 + b_9\rangle} \\
LCH_{-2} &= \frac{\ker(\partial_{-2})}{\operatorname{im}(\partial_{-1})} = \frac{\mathbb{Z}_2\langle b_{10}\rangle}{\mathbb{Z}_2\langle b_{10}\rangle} = 0
\end{aligned}$$

The Poincaré polynomial for this knot is then $3t + 2t^{-1}$ as well.

3.5 A_∞-Structure Calculation

Then, we calculated the A_∞ structure.

For Λ_1, the A_∞ structure is:

$$m_1(b_1) = a_3 + a_4 \qquad m_2(b_5, b_1) = a_3$$
$$m_1(b_2) = a_2 + a_3 \qquad m_3(b_5, b_4, b_3) = a_1$$
$$m_1(b_5) = a_1 + a_3 \qquad m_3(b_7, b_6, b_5) = a_1$$
$$m_2(b_4, b_3) = a_1 \qquad m_3(b_3, b_4, b_2) = a_2$$
$$m_2(b_3, b_4) = a_2 \qquad m_3(b_2, b_5, b_1) = a_3$$
$$m_2(b_7, b_6) = a_1 \qquad m_3(b_1, b_6, b_7) = a_4$$
$$m_2(b_6, b_7) = a_4 \qquad m_4(b_7, b_6, b_4, b_3) = a_1$$
$$m_2(b_2, b_1) = a_3 \qquad m_5(b_7, b_6, b_5, b_4, b_3) = a_1$$
$$m_2(b_2, b_5) = a_3$$

and all other m_k's are trivial. We consider the m_1 structure as our new differentials, and we have

$$LCH^*_\varepsilon = \frac{\ker(m_1)}{\operatorname{im}(m_1)} = \frac{\mathbb{Z}_2\langle a_1, a_2, a_3, a_4, b_3, b_4, b_6, b_7\rangle}{\mathbb{Z}_2\langle a_3 + a_4, a_2 + a_3, a_1 + a_3\rangle}.$$

Let $a_3 + a_4 = a_2 + a_3 = a_1 + a_3 = 0$, and we get $a_1 = a_2 = a_3 = a_4$ in $\mathbb{Z}_2$. Then, we have the cohomologies at each level:

$$LCH^1_\varepsilon = \mathbb{Z}_2\langle a_1, b_3, b_7\rangle, \quad LCH^0_\varepsilon = 0, \quad LCH^{-1}_\varepsilon = \mathbb{Z}_2\langle b_4, b_6\rangle,$$

and all other LCH^n_ε's are trivial.

Remark. *For ease of notation in the next subsection, let* $a = [a_1] = [a_2] = [a_3] = [a_4], b = [b_3], c = [b_4], d = [b_6], e = [b_7]$.

For Λ_2, the A_∞ structure is:

$$m_1(b_1) = a_2 + a_3 \qquad m_1(b_6) = b_3 \qquad m_2(b_2, b_1) = a_2$$
$$m_1(b_2) = a_1 + a_2 \qquad m_1(b_9) = b_3 \qquad m_2(b_5, b_4) = a_1$$
$$m_1(b_4) = b_3 \qquad m_1(b_{10}) = b_6 + b_9 \qquad m_2(b_8, b_7) = a_1$$

$$m_2(b_7, b_8) = a_3$$
$$m_2(b_9, b_8) = a_2$$
$$m_2(b_9, b_5) = a_3$$
$$m_2(b_6, b_5) = a_2$$
$$m_2(b_4, b_8) = a_3$$
$$m_2(b_6, b_8) = a_3$$
$$m_2(b_{10}, b_8) = a_1$$
$$m_3(b_5, b_4, b_2) = a_1$$
$$m_3(b_8, b_7, b_2) = a_1$$
$$m_3(b_2, b_9, b_8) = a_2$$
$$m_3(b_2, b_9, b_5) = a_2$$
$$m_3(b_9, b_5, b_1) = a_2$$
$$m_3(b_2, b_6, b_5) = a_2$$
$$m_3(b_6, b_5, b_1) = a_2$$
$$m_3(b_5, b_{10}, b_8) = a_2$$
$$m_3(b_1, b_7, b_8) = a_3$$
$$m_3(b_6, b_5, b_4) = b_3$$
$$m_3(b_9, b_8, b_7) = b_3$$
$$m_3(b_9, b_5, b_4) = b_3$$
$$m_3(b_{10}, b_8, b_7) = b_6$$
$$m_4(b_8, b_7, b_5, b_4) = a_1$$
$$m_4(b_3, b_5, b_{10}, b_8) = a_2$$
$$m_4(b_9, b_8, b_7, b_5) = a_2$$
$$m_4(b_2, b_9, b_5, b_1) = a_2$$
$$m_4(b_2, b_6, b_5, b_1) = a_2$$
$$m_4(b_4, b_5, b_6, b_8) = a_3$$
$$m_4(b_4, b_5, b_{10}, b_8) = b_1$$
$$m_5(b_8, b_7, b_5, b_4, b_2) = a_1$$
$$m_5(b_2, b_3, b_5, b_{10}, b_8) = a_2$$
$$m_5(b_2, b_9, b_8, b_7, b_5) = a_2$$
$$m_5(b_9, b_8, b_7, b_5, b_1) = a_2$$
$$m_5(b_9, b_8, b_7, b_5, b_4) = b_3$$

and all other m_k's are trivial. Again, we consider the m_1 structure being our new differentials, and we have

$$\begin{aligned} LCH^*_\varepsilon &= \frac{\ker(m_1)}{\operatorname{im}(m_1)} \\ &= \frac{\mathbb{Z}_2\langle a_1, a_2, a_3, b_3, b_5, b_7, b_8, b_4 + b_6, b_4 + b_9, b_6 + b_9\rangle}{\mathbb{Z}_2\langle a_2 + a_3, a_1 + a_2, b_3, b_6 + b_9\rangle} \\ &= \frac{\mathbb{Z}_2\langle a_1, a_2, a_3, b_5, b_7, b_8, b_4 + b_6, b_4 + b_9\rangle}{\mathbb{Z}_2\langle a_2 + a_3, a_1 + a_2\rangle}. \end{aligned}$$

Let $a_2 + a_3 = a_1 + a_2 = b_6 + b_9 = 0$, and we get $a_1 = a_2 = a_3$ and $b_6 = b_9$. Then, we have the cohomologies at each level:

$$LCH^1_\varepsilon = \mathbb{Z}_2\langle a_1, b_5, b_8\rangle, \quad LCH^0_\varepsilon = 0, \quad LCH^{-1}_\varepsilon = \mathbb{Z}_2\langle b_7, b_4 + b_6\rangle, \quad LCH^{-2}_\varepsilon = 0,$$

and all other LCH^n_ε's are trivial.

Remark. *Let* $a' = [a_1] = [a_2] = [a_3], b' = [b_5], c' = [b_8], d' = [b_7], e' = [b_4 + b_6] = [b_4 + b_9]$.

3.6 Cup Product and Massey Product

After calculating LCH^*_ε for both of our knots, we calculate their cup product structures. Most possible cup products are trivial, but each knot have some isomorphic nontrivial ones. For our first knot, we have the following nontrivial cup products:

$$b \cup c = [m_2(b_3, b_4)] = [a_2] = a,$$

$$c \cup b = [m_2(b_4, b_3)] = [a_1] = a,$$
$$d \cup e = [m_2(b_6, b_7)] = [a_4] = a,$$
$$e \cup d = [m_2(b_7, b_6)] = [a_1] = a.$$

Similarly, our second knot has the following nontrivial cup products:

$$b' \cup e' = a,$$
$$e' \cup b' = a,$$
$$d' \cup c' = a,$$
$$c' \cup d' = a.$$

These two cup product structures are isomorphic, so they do not distinguish our two $m(7_7)$ knots.

Our next step is to look at the Massey product structures for these two knots, to see if those might distinguish them. However, we could in fact skip this step and conclude that all of the possible Massey products for both knots would be trivial, based on facts about the level of LCH to which Massey products belong. Because a Massey product must live in a LCH at a level being the sum of the gradings of each generator (element in cohomology class which is a component of the Massey product) plus 1, and because each possible generator for our knots has either a grading of 1 or -1, then any Massey product (or odd-ordered higher-order Massey product) must have an even grading. But as mentioned above, neither knot has a nonzero element of even grading in their cohomologies. We then took the next step of considering fourth order Massey products.

3.7 Proposition about Trivial Product Structure

Since the cup product structures are the same for the two $m(7_7)$ knots, and triple Massey products are trivial on them, instead of computing the quadruple Massey product specifically, we then consider general higher-order Massey products. From the following proposition with $K = 2$, all higher-order Massey products are trivial on this pair. Therefore, product structure cannot be used to distinguish the two $m(7_7)$ knots.

Proposition 3.7.1. *Fix $K > 1$, suppose for all $k \geq K$, we have that*

$$[m_k(c_1, \cdots, c_k)] = 0 \textit{ implies } m_k(c_1, \cdots, c_k) = 0,$$

for all generators c_1 through c_k satisfying

$$\{[c_1], \cdots, [c_{k-1}]\} = 0 = \{[c_2], \cdots, [c_k]\}.$$

Then for all $k \geq K+1$, the k-th order Massey products are trivial.

Proof. Consider the k-th order Massey product given by

$$\{[c_1], \cdots, [c_k]\} = [m_k(c_1, \cdots, c_k) + m_{k-1}\text{-terms} + \cdots + m_2\text{-terms}].$$

Let $1 \leq l \leq k-2$, and let $x_{k-l,i}$ denote the i-th free variable among the (m_{k-l})-terms. We wish to show all the terms in the k-th order Massey product will eventually vanish by choosing each free variable to be 0. Now we proceed by induction on l.

For $l = 1$, we know $m_1(x_{k-1,i}) = m_2(c_i, c_{i+1})$. Since

$$\{[c_1], \cdots, [c_{k-1}]\} = 0 = \{[c_2], \cdots, [c_k]\},$$

then

$$[c_i] \cup [c_{i+1}] = 0 = [m_2(c_i, c_{i+1})],$$

and by assumption, we thus have

$$m_2(c_i, c_{i+1}) = 0,$$

so we can choose $x_{k-1,i} = 0$.

Suppose the hypothesis is true for all values less than l, consider the case for l, we have

$$m_1(x_{k-l,i}) = m_{l+1}(c_i, c_{i+1}, \cdots, c_{i+l}),$$

as other terms containing free variables of previous levels vanish by induction hypothesis. Then again, by assumption that the homology class of m_k is zero implies m_k is itself zero, we can choose $x_{k-l,i}$ to be zero.

Hence all terms defined in the k-th order Massey product eventually vanish. □

4 Conclusion

By Proposition 3.7.1, product structure cannot distinguish these two $m(7_7)$ knots. After examining all the pairs that are believed to be distinct in the atlas of [2], the two $m(7_7)$ knots form the only pair which has an augmentation; thus this is the only pair that has product structure. On other pairs of knots in the atlas of [2], product structures cannot be defined. Therefore, though there do exist knots for which product structure as an invariant is useful to distinguish them, for example, certain knots and their Legendrian mirrors in [5], product structure **cannot** be used to distinguish any pair of knots in the atlas of [2].

References

[1] Y. Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150(3):441–483, 2002.

[2] W. Chongchitmate and L. Ng. An atlas of Legendrian knots. *Exp. Math.*, 22(1):26–37, 2013.

[3] Y. Eliashberg. Invariants in contact topology. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 327–338, 1998.

[4] J. Etnyre and L. Ng. Legendrian contact homology in $\mathbb{R}^3$. *arXiv preprint arXiv:1811.10966*, 2018.

[5] J. Etnyre J. Sabloff G. Civan, P. Koprowski and A. Walker. Product structures for Legendrian contact homology. *Math. Proc. Cambridge Philos. Soc.*, 150(2):291–311, 2011.

[6] J. Swiatkowski. On the isotopy of legendrian knots. *Ann. Glob. Anal. Geom*, 10(1):195–207, 1992.