§1 Invariance

Our goal was to produce an invariant of Λ. While the homology of $(\mathcal{A}_\Lambda, \partial_\Lambda)$ is an invariant of the isotopy class of Λ, the DGA itself is not, unless we introduce a somewhat strange notion of sameness for DGAs.

Def. An <u>elementary automorphism</u> of a DGA

$$(\mathcal{A}, \partial) = (\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm}\rangle, \partial)$$

is a chain map $\phi : \mathcal{A} \to \mathcal{A}$ (i.e., $\phi \circ \partial = \partial \circ \phi$) s.t., for some $1 \leq j \leq n$,

- $\phi(a_j) = \pm t^k a_j t^\ell + u, \quad u \in \mathbb{Z}\langle a_1, \dots, \hat{a}_j, \dots, a_n, t^{\pm 1}\rangle,$
  $k, \ell \in \mathbb{Z};$
- $\phi(a_i) = a_i, \quad i \neq j;$
- $\phi(t) = t.$

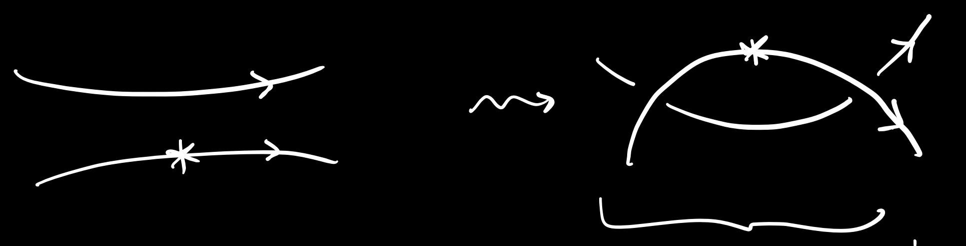
A <u>tame isomorphism</u>

$$(\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle, \partial) \longrightarrow (\mathbb{Z}\langle a_1', \dots, a_n', t^{\pm 1}\rangle, \partial')$$

is obtained by composing some # of elementary automorphisms and then $a_i \mapsto a_i'$, $t \mapsto t'$.

A tame isomorphism is a very nice isomorphism.

Problem: Sometimes a Leg. isotopy $\Lambda \rightsquigarrow \Lambda'$ has no hope of producing isomorphic DGAs.



<u>Def</u> The <u>grading k stabilization</u> of
$$(\mathbb{Z}\langle a_1, \dots, a_n, t^{\pm 1}\rangle, \partial)$$
is $\mathbb{Z}\langle e_k, e_{k-1}, a_1, \dots, a_n, t^{\pm 1}\rangle$, with $|e_k| = k$, $|e_{k-1}| = k-1$, and with ∂ extended via $\partial(e_k) = e_{k-1}$ and $\partial(e_{k-1}) = 0$.

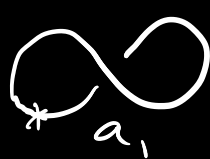
<u>Fact.</u> Stabilizing does not change cohomology.

<u>Def.</u> Call two DGAs <u>stable tame isomorphic</u> if they become tame isomorphic after some # of stabilizations.

Thm. If Λ, Λ' are Legendrian isotopic, then their DGAs $(\mathcal{A}_\Lambda, \partial_\Lambda)$, $(\mathcal{A}_{\Lambda'}, \partial_{\Lambda'})$ are stable tame isomorphic, independent of base point.

Ex. Should be clear that the double move above involves a stabilization. Pinning down the tame isomorphism is tedious, b/c new discs could be introduced.

Exercise Write down a stable tame isomorphism for



and

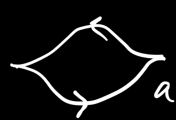
.

§2 Computing the DGA in the front projection

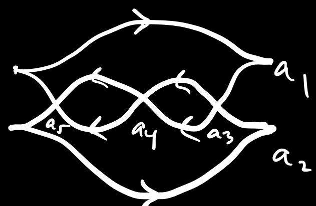
Say we have $F(\Lambda)$ w/ base point $*$ not a crossing or right cusp. Then

$$\mathcal{A}_\Lambda = \mathbb{Z}\langle \underbrace{a_1, \dots, a_n,}_{\substack{\text{crossings and} \\ \text{right cusps}}} t^{\pm 1} \rangle$$

Ex



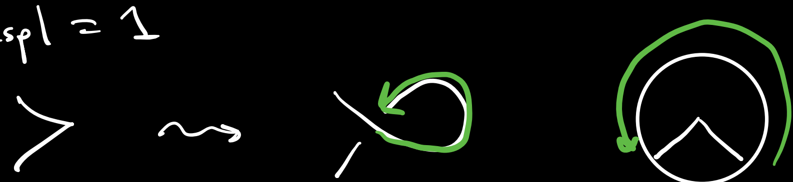
$\mathcal{A}_\Lambda = \mathbb{Z}\langle a, t^{\pm 1} \rangle$



$\mathcal{A}_\Lambda = \mathbb{Z}\langle a_1, \dots, a_5, t^{\pm 1} \rangle.$

Grading is easier than in Lagrangian projection.

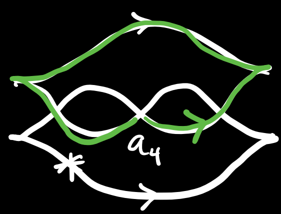
$|\text{right cusp}| = 1$



$2 \cdot \frac{3}{4} - \frac{1}{2} = 1$

$|\text{crossing}| = D(\gamma) - U(\gamma)$, γ a capping path

$\rightsquigarrow |a| = -1$

Ex 

$|a_4| = 1 - 1 = 0.$

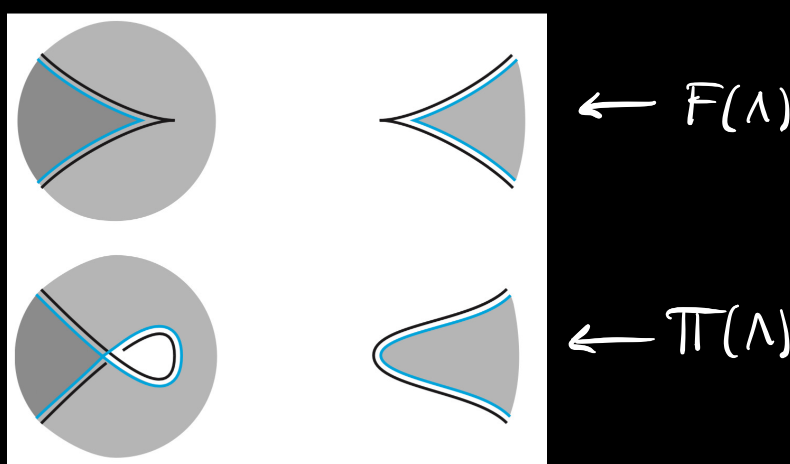
As before, ∂_Λ counts discs. A little worse to state.

We write $\Delta(a; b_1, \ldots, b_m)$ for the set of discs

$$u : (D^2_m, \partial D^2_m) \longrightarrow (\mathbb{R}^2_{xz}, F(\Lambda))$$

satisfying:

① u is an immersion, except possibly at cusps of $F(\Lambda)$, where the following is allowed:

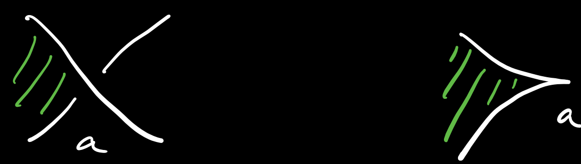


(note that this is <u>not</u> boundary puncture behavior);

② $u(\xi^0) = a$, and u sends a nbhd of ξ^0 in D^2_m to

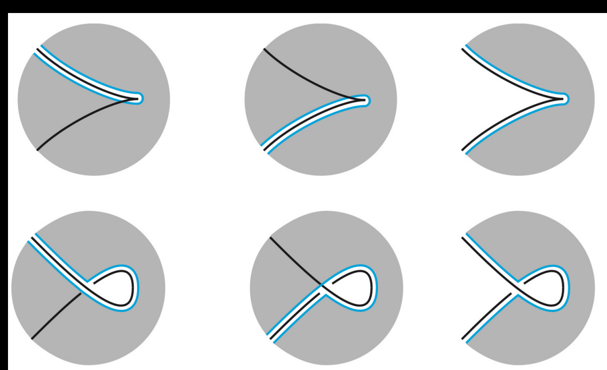
- a left-facing quadrant of a (with + Reeb sign), if a is a crossing;

- the left-facing region bounded by the cusp, if a is a right cusp;



③ $u(\xi^k) = b^k$, for $1 \le k \le n$, and u sends a nbhd of ξ^k in D^2_m to

- a quadrant of b_k w/ $-$ Reeb sign, if b_k is a crossing;
- one of the following regions, if b_k is a right cusp:



$\leftarrow F(\Lambda)$

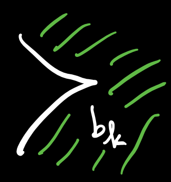
$\leftarrow \pi(\Lambda)$

Now for each $u \in \Delta(a; b_1, \ldots, b_m)$, we need a word $w(u) \in \mathcal{A}_n$ and a sign $\varepsilon(u) \in \{-1, 1\}$.

We have

$$w(u) := t^{\#\eta_0} c(b_1) t^{\#\eta_1} \cdots c(b_m) t^{\#\eta_m},$$

where $\eta_0, \eta_1, \ldots, \eta_m$ are the paths induced by ∂u, and $c(b_k) = b_k$, unless u looks like



,

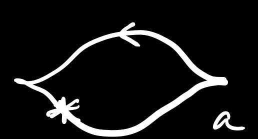
in which case $c(b_k) = b_k^2$. We also let

$d(u) :=$ # downward-facing quadrants w/ $-$ orientation sign that are covered by u

and set $\varepsilon(u) := (-1)^{d(u)}$.

Then
$$\boxed{\partial_\Lambda(a) := \begin{cases} \sum \varepsilon(u)\, w(u), & a \text{ is a crossing} \\ 1 + \sum \varepsilon(u)\, w(u), & a \text{ is a right cusp}. \end{cases}}$$

Ex

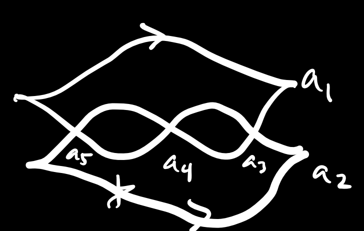


$\mathcal{A}_\Lambda = \mathbb{Z}\langle a, t^{\pm 1}\rangle$

$|a| = 1$, $|t| = 0$

$\partial_\Lambda(a) = 1 + t$

Ex



$\mathcal{A}_\Lambda = \mathbb{Z}\langle a_1, \dots, a_5, t^{\pm 1}\rangle$

$|a_1| = |a_2| = 1$

$|a_3| = |a_4| = |a_5| = |t| = 0$

$\partial_\Lambda(a_1) = 1 + a_3 + a_5 + a_5 a_4 a_3$

§3 The Chekanov–Eliashberg DGA for links

If $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_r$ is a Leg. link, we can define the DGA $(\mathcal{A}_\Lambda, \partial_\Lambda)$ in essentially the same way as before, except instead of $t^{\pm 1}$, we'll have $t_1^{\pm 1}, \ldots, t_r^{\pm 1}$.

The other key difference is the grading, b/c capping paths won't be well-defined at crossings of the two knots. Easiest to fix in $F(\Lambda)$.

A <u>Maslov potential</u> is a locally constant map

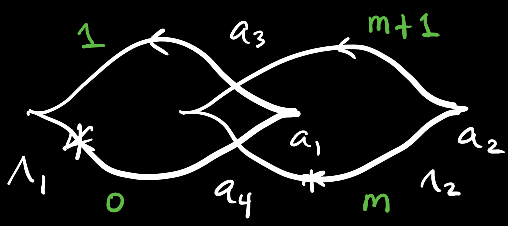
$$m: F(\Lambda) - (\text{basepoint } + \text{ cusps}) \to \mathbb{Z}$$

s.t. m incr. when we cross up cusp, decr. when we cross down cusp.

Then $|\text{cusp}| = 1$, $|\text{crossing}| = m(\text{overstrand}) - m(\text{understrand})$.

<u>Exercise</u>: Check that this jives with previous grading for knots.

<u>Note</u>. A Maslov potential is a choice, and our DGA will depend on this choice.

Ex 

$\mathcal{A}_\Lambda = \mathbb{Z}\langle a_1, a_2, a_3, a_4, t_1^{\pm 1}, t_2^{\pm 1}\rangle$

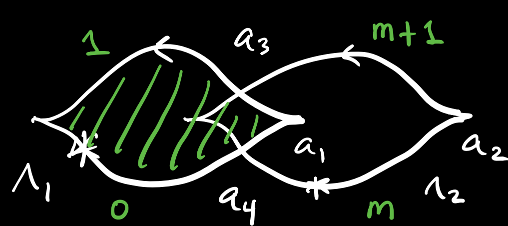
$|t_1| = -2\cdot \mathrm{rot}(\Lambda_1) = 0$

$|t_2| = -2\cdot \mathrm{rot}(\Lambda_2) = 0$

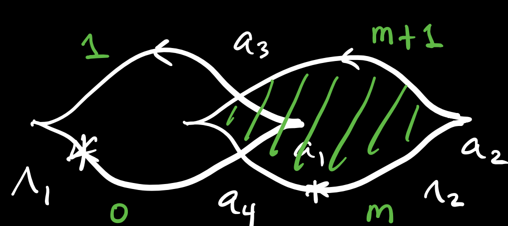
$|a_1| = |a_2| = 1$

$|a_3| = 1 - (m+1) = -m, \quad |a_4| = m - 0 = m.$

The differential is computed just as in the knot case, except that t_k counts the # of times u hits the basepoint on Λ_k.

Ex 

$\partial_\Lambda(a_1) = 1 + t_1$



$\partial_\Lambda(a_2) = 1 + t_2$

No discs for a_3, a_4, so $\partial_\Lambda(a_3) = \partial_\Lambda(a_4) = 0$.