Legendrian Connected Sums

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Introduction

Legendrian knots are knots that satisfy the condition of being everywhere tangent to the standard contact structure on $\mathbb{R}^3$. The study of Legendrian knot invariants led to the development of the Chekanov-Eliashberg different graded algebra [Che02] which is an object that can be associated to Legendrian knots. Given two Legendrian knots, there are three distinct but isotopically equivalent ways of connecting them together. In this paper we consider the relations between the Chekanov-Eliashberg differential graded algebras associated to each of the different ways of performing the connected sum.

Background

Legendrian knots and differential graded algebras of legendrian links

We can define a parameterized Legendrian knot in the standard contact 3-manifold $(\mathbb{R}^3, \xi_{std} = \ker(\mathrm{d}z - y\mathrm{d}x))$ as follows: given a regular parameterization $\Lambda(t) : S^1 \longmapsto \mathbb{R}^3$ let:

$$\dot{z}(t) - y(t)\dot{x}(t) = 0 \text{ and } \Lambda(t+1) = \Lambda(t)$$

The first condition ensures that $\dot{\Lambda}(t)$ is everywhere tangent to ξ_{std}, the second ensures that Λ is a knot.

Given two Legendrian knots Λ and Λ' we say such knots are the Legendrian isotopic if there exists a smooth family of Legendrian knots Λ_s for $s \in [0,1]$ such that $\Lambda_0 = \Lambda$ and $\Lambda_1 = \Lambda'$

Legendrian Reidemeister moves in the front projection

Given an Legendrian knot Λ, we can look at the front projection of the knot, denoted $F(\Lambda)$, from $\mathbb{R}^3$ to $\mathbb{R}^2_{xz}$, given by the map projecting (x, y, z) to (x, z). By examining the front projection of a Legendrian knot, we can recover the value of the y coordinate by $y = \frac{\mathrm{d}z}{\mathrm{d}x}$. In order to maintain the well-definedness of the y coordinate, we require that the front projection have no vertical tangencies. This is why we have right and left "cusps" in these diagrams.

From the front projection we can always determine the crossing data by noting that the strand with the more negative slope must be in front and the strand with the more positive slope in back.

We are interested in the equivalnce classes of Legendrian knots. As shown in [Świ92], two front projection represent the same Legendrian isotopy class if we can pass from one to the other by a finite sequence of Legendrian Reidemeister moves shown in Figure 1 and ambient isotopies.

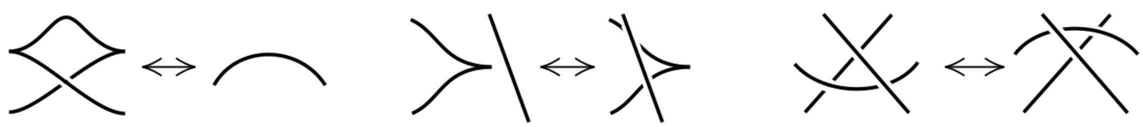

Figure 1: Legendrian Reidemeister moves

In practice is it difficult to define a Legendrian isotopy between two knots exclusively using reidemiester moves and even more difficult to show that no such isotopy exists. This leads us to examine the Chekanov-Eiashberg Differential Graded Algebra as it is used to distinguish Legendrian knots.

Differential graded algebra of Legendrian knots

The Checkanov-Eliashberg differential graded algebra is a powerful tool for distinguishing Legendrian knots. Given an orientated knot in the front projection with a base point sufficiently far from all right cusps and crossings, we define the algebra $\mathcal{A}_\Lambda$ generated over $\mathbb{Z}$ by $\{a_1, ...a_n\}$ where a_i is a crossing or a right cusp, as well as t and t^{-1} which represents the basepoint. All the right cusps are given grading 1, while the grading of the crossing a_i is determined as follows: define a capping path σ to be the path which goes from the under-strand of a_i to the over-stand and does not pass through the base-point. Then the grading a_i is $D(\sigma) - U(\sigma)$ where $D(\sigma)$ is the number of down cusps traversed by σ and $U(\sigma)$ is the number of up cusps traversed by σ.

Finally, the differential, ∂_Λ, counts the maps, u, of the unit disk D^2_n, with $n+1$ boundary points removed. We require that u is an immersion on the interior of D^2_n and that u is an immersion on the boundary of D^2_n except at cusps, there are additional restrictions on u which are outlined in [EN18].

Each function u is then given a word $w(u)$ which counts the cusps and crossings which $u(\partial D^2_n)$ interacts with as well as the orientation of ∂D^2_n as it passes through the base point. If a is a crossing then $\partial_\Lambda(a) = \Sigma\epsilon(u)w(u)$, here $\epsilon(u)$ is the product of the orientation signs of the crossing of interacted with by $w(u)$, and if a is a right cusp then we add one to the differential.

The stable-tame isomorphism class

Throughout this section, let $\mathcal{A} = (\mathbb{Z}\langle a_1, a_2, \ldots, a_n, t^{\pm}\rangle, \partial)$ and $\mathcal{A}' = (\mathbb{Z}\langle a_1', a_2,', \ldots, a_k', s^{\pm}\rangle, \partial')$ be Chekanov-Eliashberg differential graded algebras corresponding to Legendrian knots Λ, Λ' respectively.

Unfortunately, the Chekanov-Eliashberg DGA (or just the DGA for brevity) is not an invariant of Legendrian isotopy class. It is, however, an invariant of Legendrian isotopy up to stable-tame isomorphism [Che02]. We now define what this term means, starting with stabilizations.

From $\mathcal{A}$, we can define a new DGA called the ℓ-graded stabilization of $\mathcal{A}$ which we denote $\bar{\mathcal{A}}$. This new DGA has the form $\bar{\mathcal{A}} = \mathbb{Z}\langle e_1, e_0, a_1, \ldots, a_n, t^{\pm}\rangle$ where e_1, e_0 are new generators such that $|e_1| = \ell, |e_0| = \ell - 1$ and we extend ∂ to $\bar{\mathcal{A}}$ by using the Leibniz rule and declaring that $\partial(e_1) = e_0$ and $\partial(e_0) = 0$.

We now turn to the "tame" portion of stable-tame isomorphic. We start by defining an elementary automorphism. An elementary automorphism of $\mathcal{A}$ is a chain map $\varphi : \mathcal{A} \to \mathcal{A}$ such that for exactly one $j, 1 \leq j \leq n$ we have

$$\begin{aligned} \varphi(a_j) &= \pm t^k a_j t^r + w \quad && w \in \mathbb{Z}\left\langle a_1, \ldots, \hat{a_j}, \ldots, a_n, t^{\pm}\right\rangle \text{ and } k, r \in \mathbb{Z} \\ \varphi(a_i) &= a_i && i \neq j \\ \varphi(t) &= t \end{aligned}$$

In the above notation, we write $\hat{a_j}$ to mean the algebra over $\mathbb{Z}$ generated by all $a_1, \ldots, a_n, t^{\pm}$ except for a_j. Essentially, an elementary automorphism of $\mathcal{A}$ sends one generator to itself multiplied by some base point variable plus a word that does not include itself and fixes all other generators.

A tame isomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism that is the composition of some number of elementary automorphisms of $\mathcal{A}$ with a function that maps $a_i \mapsto b_i$ and $t \mapsto s$. In other words, a tame isomorphism Φ is of the form

$$\Phi = \Psi \circ \varphi_m \circ \ldots \circ \varphi_1$$

where $\varphi_1, \ldots, \varphi_m$ are elementary automorphisms of $\mathcal{A}$ and

$$\begin{aligned} \Psi : \mathcal{A} &\to \mathcal{A}' \\ a_i &\mapsto b_i \\ t &\mapsto s \end{aligned}$$

We say that two differential graded algebras are stable-tame isomorphic if they become tame isomorphic after stabilizing on one or the other or both some number of times.

Legendrian connected sums

Lengendrian connected sums provide a way to connect two distinct Legendrian knots to form one Legendrian Knots. Given two Legendrian Knots J and K we can define three different connected sums of the Legendrian knots in the front projection, as seen in Figure 2. It is known that regardless of the choice of connected sum or the choice of where the knots are connected, the resultant knots are equivalent up to Legendrian isotopy.

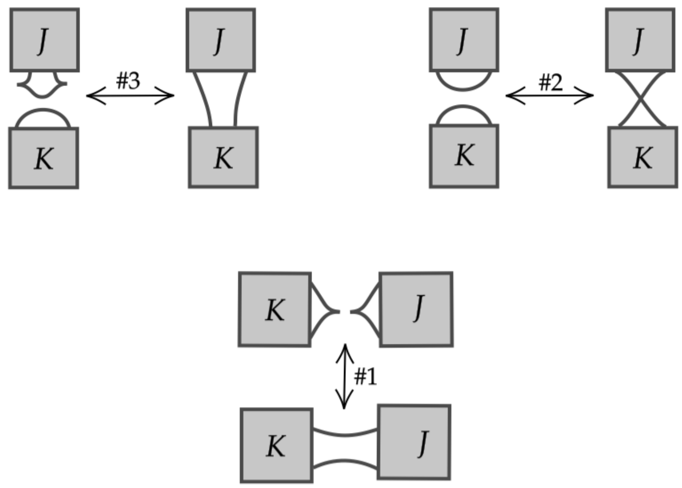

Figure 2: Three different connected sums of legendrian knots in the front projection

Research questions

Problem statement

Given two Legendrian knots J, K, there are three ways of performing a connected sum to get one knot. These are diagrammed below. It is known that Legendrian connected sum is well-defined in the sense that the three versions above yield isotopic knot. Therefore, the resulting Chekanov-Eliashberg DGAs of these connected knots should be stable-tame isomorphic. We originally set out to find the stable-tame isomorphism between the three connected sum versions, but this question proved more elusive and less well-defined than originally thought. We illustrate why below with an example.

Consider two Legendrian trefoils which we will call K and J. Suppose we want to find the Chekanov-Eliashberg DGA for the knot that we get by connecting a right cusp of K to a left cusp of J (version 1 above). There are then four distinct ways of performing this connected sum: top right cusp of K to top left cusp of J, bottom right cusp of K to top left cusp of J, bottom right cusp of K to top left cusp of J, and bottom right cusp of K to bottom left cusp of J. Figure 3 displays these four versions.

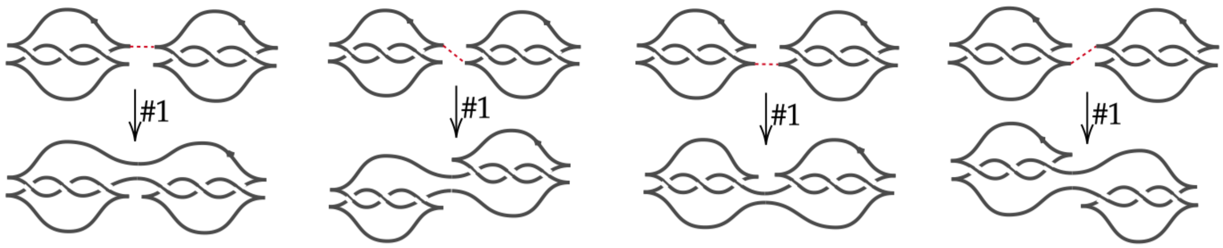

Figure 3: Different cusp connections of trefoils using #1

The four resulting knots are Legendrian isotopic, but their Chekanov-Eliashberg DGAs are not the same. Rather, they are stable-tame isomorphic. This presents a problem: if we wish to write down a general stable-tame isomorphism between the type 1 connected sum and another type of connected sum, which of version of type 1 do we choose? In Figure 4, the left two knots are formed by the version 1 of the connected sum and the right knot by a version 2. All three knots are isotopic, but the maps Φ and Ψ between their associated (stabilized) DGAs will not be the equal.

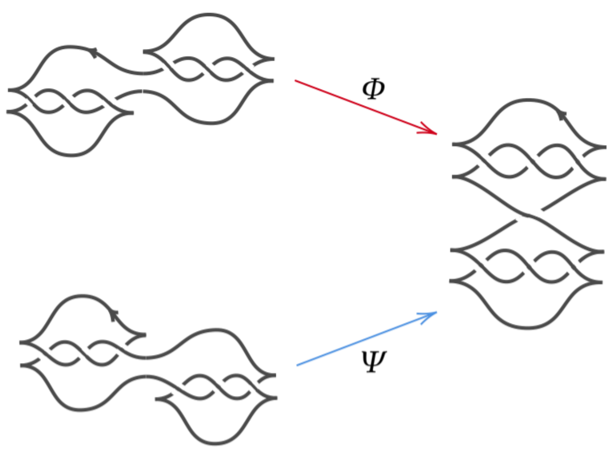

Figure 4: Maps from trefoils connected via type 1 to trefoils connected via type 2

Before one can reasonably talk about stable-tame isomorphisms between different different types of connected sums, it would be helpful to have maps between different versions of the same type. Unfortunately, we do not present a general formula for finding these in this paper. Instead, focus on the stable-tame isormophism between two Legendrian knots K, J under connected sum type 1 and type 2 with the following conditions:

- K, J are both in plat position; that is to say that in the front projection, all left cusps are aligned as $x = x_0$ and all right cusps aligned at $x = x_1$.

- K, J each have a strand at the top and bottom of the knot that connects a right cusp to a left cusp without any crossings between in the middle.
- To form the connected sum 1, we introduce a twist on the top strand of K via an R1 move and connect new left cusp to the bottom right cusp of J
- To form the connected sum 2, we connect the top strand of K to the bottom strand of J

Note that any knot can be made to satisfy the first two conditions via Reidemeister moves and ambient isotopy. By imposing these conditions, the resulting connected sums are pictorially identical up to relabeling. Figure 6 illustrates this process using two Legendrian trefoils. Also next to every knot or link in this diagram is what we will use to denote its corresponding Chekanov-Eliashberg DGA

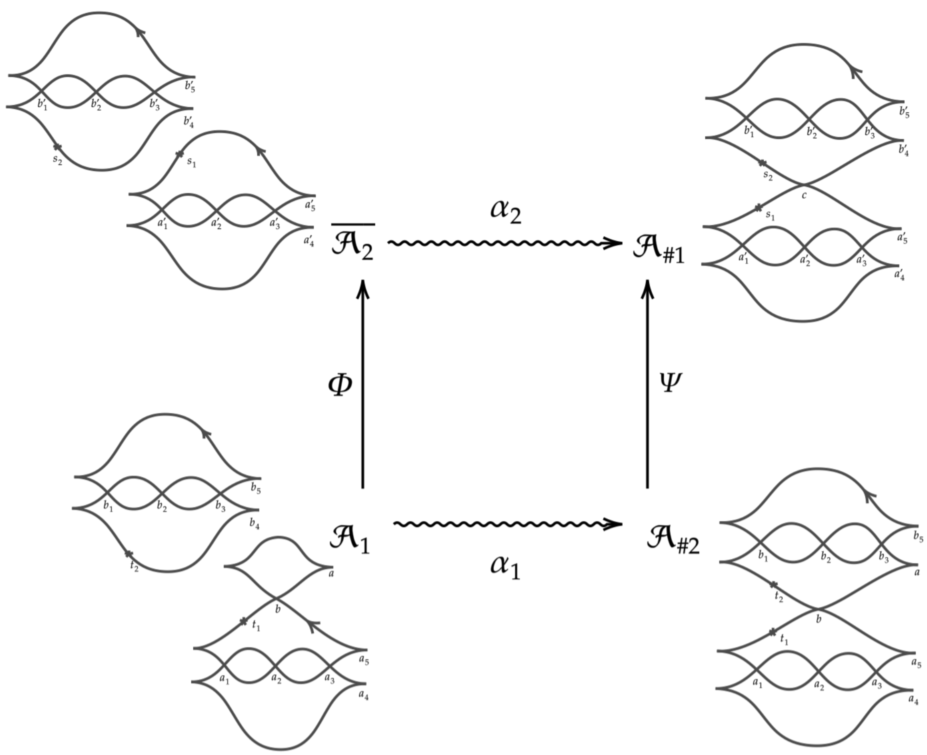

Figure 5: Legendrian trefoils connected via type 2 and type 1 respectively

We know that $\mathcal{A}_1$ and $\mathcal{A}_2$ are stable-tame isomorphic, as are $\mathcal{A}_{\#2}$ and $\mathcal{A}_{\#1}$. Our goal is to define these stable-tame isormorphisms and fill out the following diagram such that it commutes:

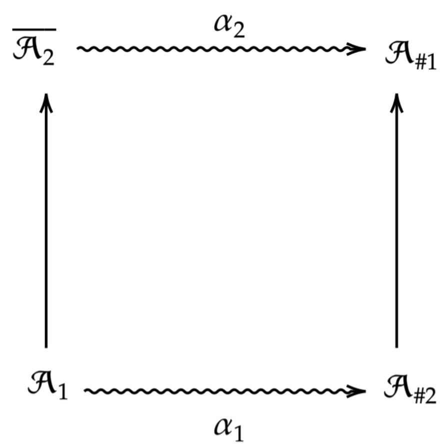

Figure 6: Commutative Diagram where Φ and Ψ are stable-tame isomorphisms and $\bar{\mathcal{A}}_2$ is the stablization of $\mathcal{A}_2$

Constructing the maps for the Legendrian trefoils

Constructing Φ

The map Φ should be a tame isomorphism from $\mathcal{A}_2$ to the 1-graded stabilization of $\mathcal{A}_1$ which we denote by $\bar{\mathcal{A}}_1$. To do this, we follow the method introduced in by Chekanov in [Che02] for Chekanov-Eliashberg DGAs over $\mathbb{Z}/2\mathbb{Z}$. We outline this method below using $\mathbb{Z}$ as our base ring.

Let Λ be a Legendrian knot, and let $(\mathcal{A}_\epsilon, \partial_\epsilon)$ be an associated Chekanov-Eliashberg DGA generated as $\mathbb{Z}\langle a_1, \ldots, a_n, t^{\pm}\rangle$. We perform a Reidemeister 1 move on Λ to obtain a new knot, Λ'. This creates two new generators: label the new right cusp a and the new crossing b. Then the Chekanov-Eliashberg DGA associated to Λ', denoted by $(\mathcal{A}_{-\epsilon}, \partial_{-\epsilon})$ is generated by $\mathbb{Z}\langle a, b, a_1, \ldots, a_n, t^{\pm}\rangle$. Finally, we will denote the stabilization of $(\mathcal{A}_\epsilon, \partial_\epsilon)$ by $(\bar{\mathcal{A}}_\epsilon, \bar{\partial}_\epsilon)$ with new generators e_1, e_0, where $|e_0| = |e_1| - 1$.

Let H denote the height function on Reed chords; that is, H takes Reed chords as arguments and outputs their heights. Without loss of generality, assume that

$$H(a_n) > H(a_{n-1}) > \ldots > H(a_1 > H(a) > H(b)$$

Consider the isomorphism $\sigma : \mathcal{A}_{-\epsilon} \to \bar{\mathcal{A}}_\epsilon$ that maps a to e_1 and b to $e_0 - 1$ and fixes all other generators. With this, we can define a new differential on $\hat{\partial}_\epsilon : \bar{\mathcal{A}}_\epsilon \to \bar{\mathcal{A}}_\epsilon$ by $\hat{\partial}_\epsilon := \sigma \circ \partial_{-\epsilon} \circ \sigma^{-1}$. Our new goal is to construct a sequence of differentials on $\bar{\mathcal{A}}_\epsilon$ such that composing them in sequence gives a tame isomorphism from $(\bar{\mathcal{A}}_\epsilon, \bar{\partial}_\epsilon)$ to $(\bar{\mathcal{A}}_\epsilon, \hat{\partial}_\epsilon)$. Composing this tame isomorphism with σ would give us the tame isomorphism we are looking for between $\bar{\mathcal{A}}_\epsilon$ and $\mathcal{A}_{-\epsilon}$.

In order to accomplish our new goal, we need to introduce a few more pieces of notation.

First, for $i = 0, 1, \ldots, n$, let $\mathcal{A}_{[i]}$ be the subalgebra of $\bar{\mathcal{A}}_\epsilon$ generated by $\mathbb{Z}\langle e_1, e_0, a_1, \ldots, a_i, t^{\pm}\rangle$. Our method for accomplishing the new goal is as follows: we find a sequence of $q_i \in \mathcal{A}_{[i]}$ and define g_i to send a_i to $a_i - q_i$ and fix all other generators. In particular, we select q_i such that $\partial_{[i]} := g_i \circ \partial[i-1] \circ g_i^{-1}$ agrees with $\bar{\partial}_\epsilon$ on $\mathcal{A}_{[i]}$. Doing this will guarantee that $\partial_{[n]}$ is the tame isomorphism we seek. We start by letting $\partial_{[0]} := \hat{\partial}_\epsilon$. It is a lemma in [Che02] that $\hat{\partial}$ (and thus $\partial_{[0]}$) agrees with $\bar{\partial}_\epsilon$ on $\mathcal{A}_{[0]}$.

We now define what q_i shall be for each i. Let $x = ye_iz \in \bar{\mathcal{A}}_\epsilon$ where y does not contain the letters e_1 or e_0 and let $h : \bar{\mathcal{A}}_\epsilon \to \bar{\mathcal{A}}_\epsilon$ a function defined by

$$\begin{aligned} h(x) &= ye_1z \quad \text{if } i = 0 \\ h(x) &= 0 \quad \text{if } i = 1 \\ h(w) &= 0 \quad \text{if } w \text{ contains no } e_i \end{aligned}$$

For each i, setting $q_i = h(\bar{\partial}_\epsilon(a_i) - \hat{\partial}_\epsilon(a_i))$ will satisfy the conditions stated previously.

The labeling of generators in $\mathcal{A}_1$ and $\mathcal{A}_2$ was chosen to be in the correct height order necessary for this process. Thus, using this algorithm, we can find the map Φ for the trefoils in figure [number]:

$$\begin{aligned} \Phi : \mathcal{A}_2 &\to \mathcal{A}_1 \\ a &\mapsto e_1 \\ b &\mapsto e_0 - 1 \\ a_i &\mapsto a'_i \quad i = 1, 2, 3, 4 \\ b_i &\mapsto b'_i \quad \text{for all } i \\ a_5 &\mapsto a'_5 + e_1ta'_3 + e_1ta'_1 + e_1ta'_1a'_2a'_3 \end{aligned}$$

Constructing Ψ

The two knots on the right hand side of figure 5 are pictorially identical up to relabeling of the vertices. This is a intentional consequence of the way that we chose to perform the connected sum 1 after a Reidemeister 1 move. Therefore, the tame isomorphism Ψ that goes from $\mathcal{A}_{\#1}$ into $\mathcal{A}_{\#2}$ has the following simple form:

$$\begin{aligned} a &\mapsto b'_4 \\ b &\mapsto c \\ a_i &\mapsto a'_i \quad \text{for } i = 1, 2, 3, 4, 5 \\ b_i &\mapsto b'_i \quad \text{for } i = 1, 2, 3, 4, 5 \\ t_i &\mapsto s_i \quad \text{for } i = 1, 2 \end{aligned}$$

Describing α_1, α_2

We were unable to find chain maps that describe what the connected sums do to the Chekanov-Eliashberg DGAs, but we can describe it using a simple algorithm. Essentially, α_1 replaces every instance of t_1 in a differential with bt_1 and replaces every instance of t_2 in a differential with t_2b. Similarly, α_2 replaces every instance of s_1 in a differential with cs_1 and every instance of s_2 in a differential with s_2c.

Further questions

In this paper, we presented maps between the connection of two trefoils using the connected sum 2 and two trefoils with a Reidemeister 1 move using the connected sum 1. The process we used should hold in more generality: the map Φ can always be found using the method presented in section 3.1, the map Ψ should always have a similar simple closed form since the resulting diagrams are the same up to relabeling, and α_1, α_2 will always behave as described if base points are chosen appropriately (near b in $\mathcal{A}_1$ and close to c in $\mathcal{A}_2$).

However, the description of α_1 and α_2 given above ignore the fact that $\bar{\mathcal{A}}_2$ and $\mathcal{A}_1$ (and similarly $\mathcal{A}_2$ and $\mathcal{A}_2$) have different number of generators. Instead, we leave it implicit that there are other changes to the DGA that must be made when perfoming the connected sum (i.e. adding and removing a generator) that are not captured by α_1 and α_2. One reason for this is because α_2 in particular describes a change to $\bar{\mathcal{A}}_2$ rather than $\mathcal{A}_2$. We suspect that there may exist a type of projection chain mapping $\varphi : \bar{\mathcal{A}}_2 \to \mathcal{A}_2$ that comes into play. A more fundamental issue is that α_1 and α_2 as we have described them are not maps. The primary point of extension that we see is finding proper chain maps to describe the affect of performing the connected sum on knots.

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