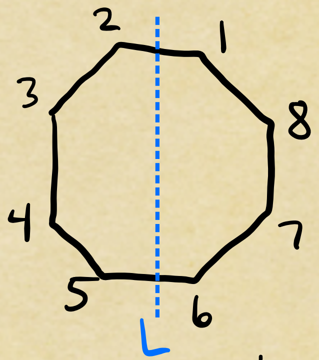
Day 6 June 5, 2024

For any $n \geq 3$, we define the n^{th} dihedral group D_n to be the symmetry group of a regular n-gon.



By labeling the vertices of our n-gon, we can realize D_n as a subgroup of S_n.

CCW rotation $= (12345678)$

refl across $L = (12)(38)(47)(56)$

$|D_n| = 2n$

We have n possible images for vertex 1.

Once $\sigma(1)$ is known, 2 possibilities for $\sigma(2)$.

$\sigma(1)$ & $\sigma(2)$ then determine all other positions.

$r =$ CCW rotation by one notch
$= (1234 \cdots n)$

$s =$ reflection across the line thru vertex 1 and the center

Then $r^n = (1)$, $s^2 = (1)$, $(sr)^2 = (1)$.

In fact,

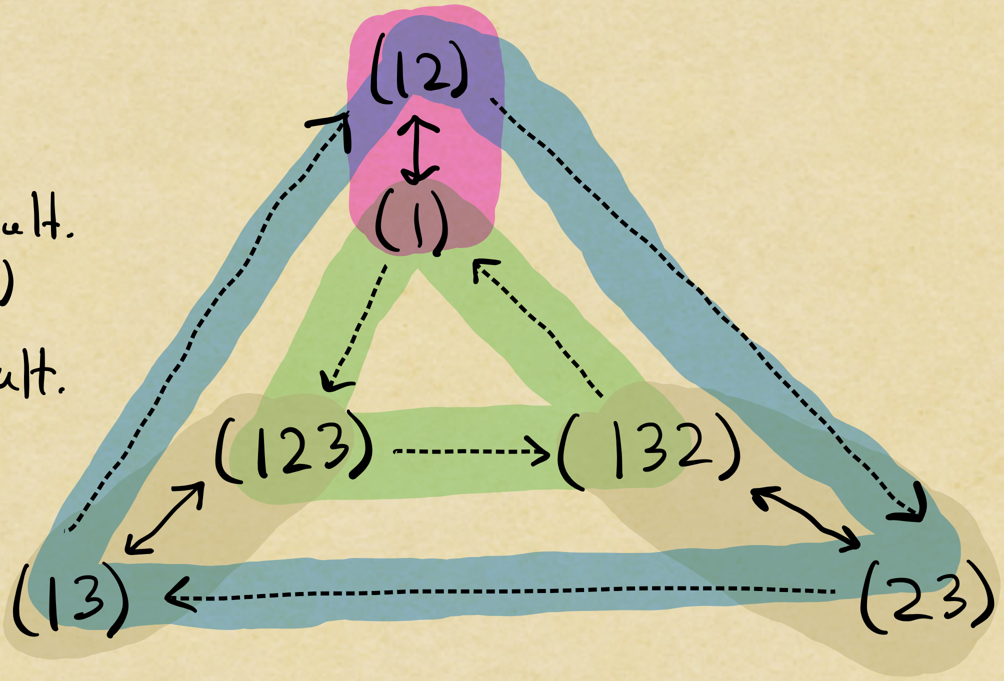
$$D_n = \langle r, s \mid r^n = s^2 = (sr)^2 = (1) \rangle.$$

Cosets

$D_3 = S_3$

$\dashrightarrow$ = right mult. by (123)

$\longleftrightarrow$ = right mult. by (12)



$(12)(123) = (23)$
$(23)(123) = (13)$
$(13)(123) = (12)$

$(123)(12) = (13)$
$(13)(12) = (123)$

$H = \langle (12) \rangle$
$K = \langle (123) \rangle$

Two elements lie in the same gold set iff their "difference" is in H.

Any two elements in the blue set have a "difference" which lies in K.

Def. Let $H \leq G$. The left coset of H with representative $g \in G$ is the set

$$gH := \{gh \mid h \in H\}.$$

The right coset of H with representative $g \in G$ is the set

$$Hg := \{hg \mid h \in H\}.$$

Ex. $G = D_3$, $H = \langle (12) \rangle$, $K = \langle (123) \rangle$.

$$\left.\begin{array}{l}(1)H = \{(1)(1), (1)(12)\} = \{(1), (12)\} \\ (12)H = \{(12)(1), (12)(12)\} = \{(12), (1)\}\end{array}\right\rangle H$$

$(123)H = \{(123)(1), (123)(12)\} = \{(123), (13)\}$

$(13)H = \{(13)(1), (13)(12)\} = \{(13), (123)\}$

(123) & (13) rep. same left coset

$(132)H = \{(132)(1), (132)(12)\} = \{(132), (23)\}$

$(23)H = \{(23)(1), (23)(12)\} = \{(23), (132)\}$

(132) & (23) rep. same left coset.

Exercise

1. Compute the left cosets of K.
2. Check that the right cosets of H do not coincide with the left cosets.
3. What about those of K?

Lemma. Let $H \leq G$ and pick $g_1, g_2 \in G$. Then TFAE:

1. $g_1 H = g_2 H$;
2. $H g_1^{-1} = H g_2^{-1}$;
3. $g_1 H \subseteq g_2 H$;
4. $g_2 \in g_1 H$;
5. $g_1^{-1} g_2 \in H$.

Rmk. Condition 5 captures the idea that $g_1, g_2 \in G$ rep. the same (left) coset of H iff their "difference" is in H.

$$g_1^{-1} g_2 \in H \Rightarrow g_1^{-1} \circ g_2 = h, \text{ for some } h \in H$$
$$\Rightarrow g_2 = g_1 \circ h.$$

So g_2 is the same symmetry as g_1, if we're willing to ignore the symmetries of H.

Ex. $G = \mathbb{Z}$. $g_1^{-1} g_2 = (-g_1) + g_2 = g_2 - g_1$.

Thm. Let $H \leq G$ be a subgroup of a group G. Then the left (respectively, right) cosets of H partition G.

(Proof.) Certainly every $g \in G$ lies in a left coset of H: $g \in gH$.

So we NTS that if $g_1 H \cap g_2 H \neq \emptyset$, then in fact $g_1 H = g_2 H$.

§ $g_1 H \cap g_2 H \neq \emptyset$. Pick $a \in g_1 H \cap g_2 H$.
Then $a = g_1 h_1 = g_2 h_2$, for some $h_1, h_2 \in H$.

$$\downarrow$$

$$g_2 = g_1 h_1 h_2^{-1} \in g_1 H.$$

By condition 4 of lemma, $g_1 H = g_2 H$. ∎