Day 18 June 27, 2024

**Recall:** If $H \leq G$, then $|H|$ divides $|G|$.

Suppose $m$ is an integer which divides $|G|$.
Must $G$ admit $H \leq G$ with $|H| = m$? **No.** (e.g., $6 \mid |A_4|$)

> **Thm** [First Sylow Theorem]
> If $p$ is prime, $r$ is a nonnegative integer, and $|G|$ is divisible by $p^r$, then $G$ admits a subgroup of order $p^r$.

(Proof.) We'll use strong induction on $|G|$.

Case $r = 0$: $\{e\} \leq G$ ✓

Assume $r \geq 1$.

Base case: $|G| = p^r$. Let $H = G$. ✓

Inductive hypothesis: Any group with order smaller than $|G|$ and divisible by $p^r$ admits a subgroup of order $p^r$.

Consider the class equation:

$$|G| = |Z(G)| + \sum_{i=1}^{\ell} [G : C_G(g_i)].$$

Case ①: $\exists\ 1 \leq i \leq \ell$ s.t. $p \nmid [G : C_G(g_i)]$

Since

$$|C_G(g_i)| = \frac{|G|}{[G : C_G(g_i)]},$$

$|C_G(g_i)| < |G|$ is divisible by $p^r$.

So $\exists\ H \leq C_G(g_i)$ with $|H| = p^r$.

$\therefore H \leq G$ with $|H| = p^r$ and we're finished.

Case 2: $\forall\ 1 \leq i \leq \ell$, $p \mid [G : C_G(g_i)]$.

Reduce the class equation mod $p$:

$$0 \equiv |Z(G)| + 0 \pmod{p}.$$

But $e \in Z(G)$, so $p \mid |Z(G)|$.

Since $Z(G)$ is abelian, our abelian version of Cauchy's theorem gives $N \leq Z(G)$ of order $p$.

Since $N \leq Z(G)$, $N$ is normal in $G$, so we consider $G/N$.

$|G/N| < |G|$ & $p^{r-1} \mid |G/N|$

Inductive hyp $\Rightarrow$ $K \leq G/N$ with $|K| = p^{r-1}$.

Correspondence thm $\Rightarrow$ $H := \phi^{-1}(K)$ is a subgroup of $G$ containing $N$.

Check: $|H| = p^r$. ∎

**Def.** A Sylow $p$-subgroup of a group $G$ is a subgroup of order $p^n$, where $|G| = p^n m$ and $\gcd(p, m) = 1$.

## A group acts on its subgroups

Let $\mathcal{G} = \{\text{subgroups of } G\}$. Then $G \circlearrowright \mathcal{G}$ by

$$g.H := gHg^{-1}, \quad \forall g \in G \ \&\ H \in \mathcal{G}.$$

## Fixed points = normal subgroups

**Def.** The normalizer of $H \in \mathcal{G}$, denoted $N(H)$, is its stabilizer subgroup.

i.e.,

$$N(H) := \{g \in G \mid gHg^{-1} = H\}.$$

### Exercises

1. For any $H \leq G$, $N(H)$ is the largest subgroup of $G$ containing $H$ as a normal subgroup.
2. Let $X$ be a $G$-set and $H \leq G$. Show that each orbit $\mathcal{O}_x \subseteq X$ is an $H$-set.
3. Let $H \leq G$ and $K \in \mathcal{G}$. Then
$$|\mathcal{O}_K^H| = [H : N(K) \cap H],$$
where $\mathcal{O}_K^H := \{h.K \mid h \in H\}$.
4. If $P \leq G$ is a Sylow $p$-subgroup and $x \in N(P)$ has order $|x| = p^k$, then $x \in P$.

## Counting Sylow p-subgroups

**Thm.** [Second Sylow Theorem]
For any prime $p$ and finite group $G$, the Sylow $p$-subgroups of $G$ are pairwise conjugate.

i.e., $\mathcal{O}_p = \{\text{Sylow } p\text{-subgroups}\}$

**Thm.** [Third Sylow Theorem]
Let $G$ be a finite group and let $p$ be a prime s.t. $p \mid |G|$. If $n_p$ denotes the number of Sylow $p$-subgroups of $G$, then

1. $n_p \equiv 1 \pmod{p}$;
2. $n_p$ divides $|G|$.

## Applications

1. Classify all groups of order 99.

$$99 = 3^2 \cdot 11$$

$n_3 = \#$ of Sylow 3-subgroups

$n_3 = 3k+1$, for some $k \geq 0$
Also, $n_3 \mid 99$.

$1, \cancel{4}, \cancel{7}, \cancel{10}, \cancel{13}, \ldots$ $\qquad n_3 = 1.$

$n_{11} = 11k+1$, for some $k \geq 0$
$n_{11} \mid 99$.

$1, \cancel{12}, \cancel{23}, \ldots$ $\qquad n_{11} = 1$

Let $H$ = unique Sylow 3-group
$K$ = unique Sylow 11-group.
By $2^{nd}$ Sylow, these are normal.

$H \cap K = \{e\}$, since $\gcd(|H|, |K|) = 1$.

Normal $\Rightarrow hk = kh$, $\forall\, h \in H$ & $k \in K$.

$\therefore HK \cong H \times K$

$|H \times K| = 9 \cdot 11 = 99$, so $HK = G$.

So $G \cong H \times K$.

$|K| = 11 \Rightarrow K \cong \mathbb{Z}_{11}$.

$|H| = 9 = 3^2 \Rightarrow H$ is abelian

FTFGAG $\Rightarrow H \cong \mathbb{Z}_9$ or $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$

$\therefore G \cong \mathbb{Z}_9 \times \mathbb{Z}_{11}$ OR $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11}$.

2 __Thm__. If $G$ is a group of order $pq$, with $p$ and $q$ distinct primes, then $G$ is not simple. Moreover, if $q \not\equiv 1 \pmod{p}$, then $G$ is cyclic.