Day 17 June 26, 2024

Recall:

- An action of G on X is a homom. $G \to S_X$.
- The orbit of $x \in X$ is $\mathcal{O}_x = \{g.x \mid g \in G\}$.
- The fixed point set of $g \in G$ is $X_g = \{x \in X \mid g.x = x\}$.
- The stabilizer subgroup of $x \in X$ is $G_x = \{g \in G \mid g.x = x\}$.

Thm. Let G be a finite group and let X be a finite G-set. Then $|\mathcal{O}_x| = [G:G_x]$, $\forall x \in X$.

(Proof.) We'll establish a bijection $\phi: \mathcal{O}_x \to \mathcal{L}_{G_x}$, where $\mathcal{L}_{G_x}$ is the set of left cosets of $G_x \leq G$.

Define $\phi(y) := gG_x$, where $g.x = y$.

Well-def: $\mathcal{S}$ $g_1.x = g_2.x$.

Then $x = g_1^{-1}.(g_2.x) = (g_1^{-1}g_2).x$, so $g_1^{-1}g_2 \in G_x$.

$\therefore$ $g_1G_x = g_2G_x$, so $\phi(g_1.x) = \phi(g_2.x)$. ✓

Surjective: $\forall$ $gG_x \in \mathcal{L}_x$, $\phi(g.x) = gG_x$ ✓

Injective: $\mathcal{S}$ $\phi(y_1) = \phi(y_2)$.

Then $g_1G_x = g_2G_x$, where $y_1 = g_1.x$ & $y_2 = g_2.x$.

So $g_1^{-1}g_2 \in G_x \Rightarrow (g_1^{-1}g_2).x = x$

$\Rightarrow g_1.((g_1^{-1}g_2).x) = g_1.x$

$\Rightarrow g_2.x = g_1.x$

$\Rightarrow y_2 = y_1$. ✓

∎

Thm [The class equation]

Let G be a finite group with r distinct conjugacy classes, and let $g_1, g_2, \dots, g_r$ be representatives of the conjugacy classes.

Then

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)],$$

where $Z(G)$ is the center of G.

(Proof.) For any G-set X, we can define

$$X_G := \{x \in X \mid g.x = x, \ \forall g \in G\}.$$

$\forall \, x \in X_G$, $\mathcal{O}_x = \{x\}$. Because the orbits partition X,

$$|X| = |X_G| + \sum_{i=1}^{r} |\mathcal{O}_{x_i}|,$$

if X is finite, where $x_1, \dots, x_r$ are rep's of the nontrivial orbits.

In the case of the conjugation action, $X_G = Z(G)$ and nontrivial orbits are conjugacy classes.

So

$$\begin{aligned} |G| &= |Z(G)| + \sum_{i=1}^{r} |\mathcal{O}_{g_i}| \\ &= |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)]. \end{aligned}$$

prev. thm.

∎

Applications of the class equation

Thm. Any group of prime-power order has nontrivial center.

(Proof.) § $|G| = p^n$, with p prime, and consider the class equation

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$

According to Lagrange's theorem, each index $[G : C_G(g_i)]$ must divide p^n. Since these are not equal to 1, they must each be divisible by p. Reducing the class equation modulo p gives

$$0 \equiv |Z(G)| + 0 \pmod{p}.$$

Since $e \in Z(G)$, $|Z(G)| \neq 0$. So $p \mid |Z(G)|$ and, in particular, $|Z(G)| > 1$. $\blacksquare$

Cor. Any group of order p^2, with p prime, is abelian.
(Proof.) Exercise. $\blacksquare$

<u>Thm</u>. [Burnside's counting theorem]
Let G be a finite group, X a G-set, and k the number of orbits of X. Then

$$k = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

In words: The average # of points in X fixed by an element of G is equal to the # of orbits.

(Proof.)

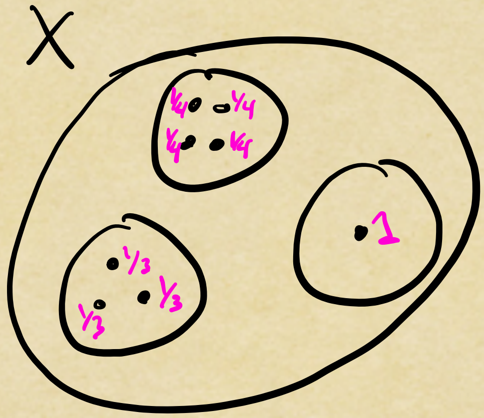
$$\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{|G|} \sum_{g \in G} \left| \{ x \in X \mid g.x = x \} \right|$$
$$= \frac{1}{|G|} \left| \{ (x, g) \in X \times G \mid g.x = x \} \right|$$
$$= \frac{1}{|G|} \sum_{x \in X} \left| \{ g \in G \mid g.x = x \} \right|$$
$$= \frac{1}{|G|} \sum_{x \in X} |G_x|.$$

Recall that Lagrange gives $[G : G_x] = |G| / |G_x|$,
So $|G_x| = |G| / [G : G_x]$.
Then

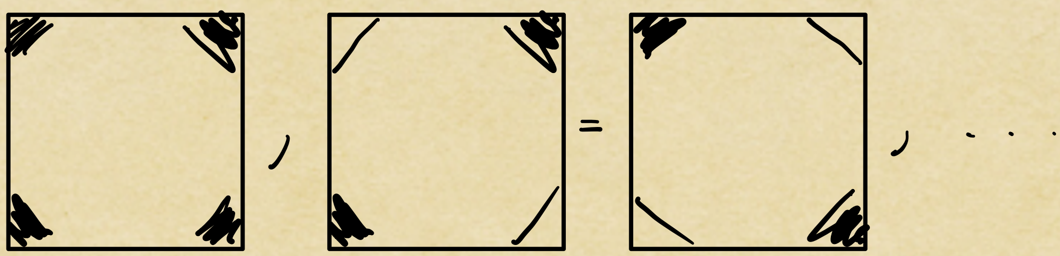
$$\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{|G|} \sum_{x \in X} \frac{|G|}{[G : G_x]} = \sum_{x \in X} \frac{1}{[G : G_x]}.$$

By theorem above, $[G : G_x] = |\mathcal{O}_x|$, so

$$\frac{1}{|G|} \sum_{g \in G} |X_g| = \sum_{x \in X} \frac{1}{|\mathcal{O}_x|} = k. \qquad \blacksquare$$



Q. In how many ways can we color the vertices of a square black or white?



Let X = colorings of the square in a fixed pos.

$$= \left\{ \begin{array}{cccc} BBBB & BBBW & BBWB & BWBB \\ WBBB & BBWW & BWBW & WBBW \\ WWBB & WBWB & BWWB & BWWW \\ WBWW & WWBW & WWWB & WWWW \end{array} \right\}$$

Let $G = D_4$, with

$e = (1)$ $\quad r = (1234)$ $\quad r^2 = (13)(24)$ $\quad r^3 = (1432)$

$s = (12)(34)$ $\quad sr = (24)$ $\quad sr^2 = (14)(23)$ $\quad sr^3 = (13)$.

Notice that $G \circlearrowright X$.

Compute each X_g:

16 $X_e = X$

2 $X_r = \{BBBB, WWWW\}$

4 $X_{r^2} = \{BBBB, WWWW, BWBW, WBWB\}$

2 $X_{r^3} = \{BBBB, WWWW\}$

4 $X_s = \{BBBB, WWWW, BBWW, WWBB\}$

8 $X_{sr} = \{BBBB, WWWW, BWBW, WBWB, BWWW, WBBB, BBWB, WWBW\}$

4 $X_{sr^2} = \{BBBB, WWWW, BWWB, WBBW\}$

8 $X_{sr^3} = \{BBBB, WWWW, BWBW, WBWB, WBWW, BWBB, BBBW, WWWB\}$

$$k = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

$$= \frac{1}{8}(16 + 2 + 4 + 2 + 4 + 8 + 4 + 8)$$

$$= \frac{48}{8} = 6.$$