Day 16 June 25, 2024

Def. We call a group G solvable if $\exists$ a subnormal series $(H_i)_{i=0}^{n}$ of G s.t. H_i/H_{i-1} is abelian, for $1 \leq i \leq n$.

Ex S_3 is solvable:

$$\underset{\mathbb{Z}_3}{1 \trianglelefteq \langle (1\,2\,3) \rangle} \underset{\mathbb{Z}_2}{\trianglelefteq S_3}$$

So is S_4:

$$\underset{\mathbb{Z}_2 \times \mathbb{Z}_2}{1 \trianglelefteq \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle} \underset{\mathbb{Z}_3}{\trianglelefteq A_4} \underset{\mathbb{Z}_2}{\trianglelefteq S_4}.$$

Prop. For $n \geq 5$, S_n is not solvable.

"Proof." The following is a composition series for S_n:

$$1 \trianglelefteq A_n \trianglelefteq S_n,$$

provided $n \geq 5$. (The book has a proof that A_n is simple for $n \geq 5$.) Given any subnormal series of S_n, we can refine to get a composition series. By Jordan-Hölder, this comp. series will be isomorphic to $1 \trianglelefteq A_n \trianglelefteq S_n$, so its quotients will be A_n & $\mathbb{Z}_2$. Since A_n is not abelian, S_n is not solvable. ∎

Group actions

Def For any set X and any group G, a (left) action of G on X is a homomorphism $G \to S_X$.

Given an action of G on X, we call X a G-set.

Rmk. An action of G upon X may be realized as a map
$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g.x$$
s.t. ① $e.x = x$, $\forall\, x \in X$;
② $(g_1 g_2).x = g_1.(g_2.x)$, $\forall\, g_1, g_2 \in G$ & $x \in X$.

Ex.
① $G = GL_n(\mathbb{R}) \circlearrowright X = \mathbb{R}^n$, as does any subgroup of G.
$$G \longrightarrow S_X$$
$$A \longmapsto (\underline{\vec{x} \mapsto A\vec{x}})$$
the transformation rep'd by A in the std. basis

② $D_n \circlearrowright X = \{1, 2, \ldots, n\}$.
Label the vertices of a regular n-gon $1, 2, \ldots, n$ and use this to obtain $D_n \longrightarrow S_X = S_n$.

③ Any group G acts upon itself by conjugation. The homom. $G \longrightarrow S_G$ is
$$g \mapsto (x \mapsto gxg^{-1}).$$
this is an elt. of G calling it x to emphasize that it's being acted upon.

④ $$G \longrightarrow S_G$$
$$g \mapsto (x \mapsto gx)$$
is the left regular representation

Def. Given $G \curvearrowright X$, we call $x_1 \in X$ and $x_2 \in X$ **G-equivalent**, written $x_1 \sim_G x_2$, if $\exists\, g \in G$ s.t.
$$g \cdot x_1 = x_2.$$

Prop. For any G-set X, G-equivalence is an equivalence relation on X.

(Proof.) Exercise. $\blacksquare$

Def. Let X be a G-set.

1. The **orbits** of X under G are the G-equiv. classes, with the orbit containing x denoted $\mathcal{O}_x$.
2. The **fixed point set** of a given $g \in G$ is
$$X_g := \{x \in X \mid g.x = x\}.$$
3. The **stabilizer subgroup** or **isotropy subgroup** of a fixed $x \in X$ is
$$G_x := \{g \in G \mid g.x = x\}.$$

Prop. For any $G \curvearrowright X$ and any $x \in X$, $G_x \leq G$.

(Proof.) Exercise. $\blacksquare$

Ex.

1. $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$.

   The orbits are $\mathbb{R}^n - \{\vec{0}\}$ and $\{\vec{0}\}$.

   Given $A \in GL_n(\mathbb{R})$, $X_A = \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{v}\}$ is the eigenspace of A with eigenvalue 1.
   Most matrices will have $X_A = \{\vec{0}\}$.

Notice that $X_I = \mathbb{R}^n$.

Given $\vec{v} \in \mathbb{R}^n$, $G_{\vec{v}} = \{A \in GL_n(\mathbb{R}) \mid A\vec{v} = \vec{v}\}$.

$G_{\vec{0}} = GL_n(\mathbb{R})$

For $\vec{v} \neq \vec{0}$, $G_{\vec{v}}$ is isomorphic to

$$(\mathbb{R}^{n-1} \times GL_{n-1}(\mathbb{R}), *),$$

where $(\vec{v}, A) * (\vec{w}, B) = (\vec{v} + A\vec{w}, AB)$.

② $D_n \curvearrowright X = \{1, 2, \ldots, n\}$.

Only one orbit – all of X.

Given $\sigma \in D_n$, $X_\sigma = \{k \in X \mid \sigma . k = k\}$.

$X_e = X$, $X_{r^k} = \emptyset$, X_s = vertices thru which axis passes

Given $k \in X$, $G_k =$

③ For G acting upon itself by conjugation, the nontrivial orbits (i.e., those with more than one element) are called the conjugacy classes of G.

For $x \in G$, G_x is called the centralizer subgroup, denoted $C_G(x)$.

For any $g \in G$, $X_g = \{x \in X \mid gxg^{-1} = x\}$
$= \{x \in X \mid gx = xg\} = C_G(g)$.