Day 15 June 24, 2024

Recall:

<u>Def.</u> A **subnormal series** of (or for) a group G is a finite sequence of nested subgroups

$$\{e\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_{n-1} \leq H_n = G$$

s.t. H_{i-1} is normal as a subgroup of H_i, $1 \leq i \leq n$.

The **length** of a subnormal series is the number of proper inclusions.

We call $(K_j)_{j=0}^{m}$ a **refinement** of $(H_i)_{i=0}^{n}$ if each H_i appears as some K_j.

We call a subnormal series a **composition series** if each quotient H_i/H_{i-1} is simple.

<u>Def.</u> We say that the subnormal series $(H_i)_{i=0}^{n}$ and $(K_j)_{j=0}^{m}$ of a group G are **isomorphic** if there's a bijective correspondence between the collections of quotients $\{H_i/H_{i-1} \mid 1 \leq i \leq n\}$ & $\{K_j/K_{j-1} \mid 1 \leq j \leq m\}$.

<u>Ex.</u>

$$\langle 0 \rangle \overset{\mathbb{Z}_2}{\leq} \langle 15 \rangle \overset{\mathbb{Z}_3}{\leq} \langle 5 \rangle \overset{\mathbb{Z}_5}{\leq} \mathbb{Z}_{30}$$

& $$\langle 0 \rangle \underset{\mathbb{Z}_2}{\leq} \langle 15 \rangle \underset{\mathbb{Z}_5}{\leq} \langle 3 \rangle \underset{\mathbb{Z}_3}{\leq} \mathbb{Z}_{30}$$

are isomorphic subnormal series (in fact they're composition series)

Prop. Every finite group admits a composition series.

(Proof.) If G is simple, then $\{e\} \trianglelefteq G$ is a composition series.

If G is not simple, $\exists$ a subnormal series

$$\{e\} \leq H_1 \leq H_2 \leq \cdots \leq H_n \leq G.$$

If each quotient H_i/H_{i-1} is simple, we're done. If some H_i/H_{i-1} is not simple, $\exists\ N' \trianglelefteq H_i/H_{i-1}$, so the correspondence thm gives

$$H_{i-1} \subsetneq N \subsetneq H_i.$$

So consider the subnormal series

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{i-1} \leq N \leq H_i \leq \cdots \leq H_n \leq G.$$

Continue inductively.

B/c G is finite, this process will terminate. ∎

Thm. [Jordan–Hölder]
Any two composition series of a fixed group G are isomorphic.

(Proof.) $\S$ $(H_i)_{i=0}^{n}$ & $(K_j)_{j=0}^{m}$ are comp. series of a group G and that $n \leq m$.

We will argue by induction on n.

If $n=1$, G is simple, so $\{e\} \leq G$ is the only subnormal series of G, so we're finished.

For the inductive step, we assume the conclusion for any group admitting a comp. series of length less than n.

Note: If $H_{n-1} = K_{m-1}$, then the I.H. takes care of everything, b/c $(H_i)_{i=0}^{n-1}$ and $(K_j)_{j=0}^{m-1}$ are then isomorphic.

Now § $H_{n-1} \neq K_{m-1}$. We'll build two comp. series for $H_{n-1} \cap K_{m-1}$.
Consider the subnormal series

$$\{e\} = H_0 \cap K_{m-1} \leq H_1 \cap K_{m-1} \leq \cdots \leq H_{n-1} \cap K_{m-1} \quad (A)$$

and

$$\{e\} = H_{n-1} \cap K_0 \leq H_{n-1} \cap K_1 \leq \cdots \leq H_{n-1} \cap K_{m-1}, \quad (B)$$

where we've removed any nonproper inclusions.

<u>Claim:</u> (A) & (B) are comp. series

Consider

$$\frac{H}{H \cap K} \cong \frac{HK}{K}$$

$$(H_i \cap K_{m-1})/(H_{i-1} \cap K_{m-1})$$

$$= (H_i \cap K_{m-1})/(H_{i-1} \cap (H_i \cap K_{m-1}))$$

2nd I.T.

$$\cong H_{i-1}(H_i \cap K_{m-1})/H_{i-1} \trianglelefteq H_i/H_{i-1}.$$

So $(H_i \cap K_{m-1})/(H_{i-1} \cap K_{m-1})$ is isomorphic to a normal subgroup of H_i/H_{i-1}.
So $(H_i \cap K_{m-1})/(H_{i-1} \cap K_{m-1})$ is isomorphic to either H_i/H_{i-1} or $\{e\}$. Either way, it's simple.
$\therefore$ (A) is a comp. series. Same goes for (B).

By the I.H., (A) & (B) are isomorphic.

From (A) we can obtain a comp. series for H_{n-1}:

$$\{e\} = H_0 \cap K_{m-1} \leq H_1 \cap K_{m-1} \leq \cdots \leq H_{n-1} \cap K_{m-1} \leq H_{n-1}.$$

By I.H., this is isomorphic to $(H_i)_{i=0}^{n-1}$.
Similarly,

$$\{e\} = H_{n-1} \cap K_0 \leq H_{n-1} \cap K_1 \leq \cdots \leq H_{n-1} \cap K_{m-1} \leq K_{m-1}$$

is isomorphic to $(K_j)_{j=0}^{m-1}$.

So

$$\{e\} = H_0 \cap K_{m-1} \leq H_1 \cap K_{m-1} \leq \cdots \leq H_{n-1} \cap K_{m-1} \leq H_{n-1} \leq G \quad (C)$$

is isomorphic to $(H_i)_{i=0}^{n}$ and

$$\{e\} = H_{n-1} \cap K_0 \leq H_{n-1} \cap K_1 \leq \cdots \leq H_{n-1} \cap K_{m-1} \leq K_{m-1} \leq G \quad (D)$$

is isomorphic to $(K_j)_{j=0}^{m}$.

We'll now show that (C) & (D) are isomorphic.

B/c (A) & (B) are isomorphic, we NTS

$$\{H_{n-1}/(H_{n-1} \cap K_{m-1}),\ G/H_{n-1}\}$$

and

$$\{K_{m-1}/(H_{n-1} \cap K_{m-1}),\ G/K_{m-1}\}$$

are equivalent.

Recall: $H_{n-1} \neq K_{m-1}$. $\therefore$ $H_{n-1}K_{m-1}$ is a normal subgroup of G which properly contains H_{n-1}.

So $H_{n-1}K_{m-1}/H_{n-1}$ is a nontrivial normal subgroup of G/H_{n-1}, which is simple.

$$\therefore\ H_{n-1}K_{m-1}/H_{n-1} = G/H_{n-1}$$

$$\therefore\ H_{n-1}K_{m-1} = G$$

2^{nd} I.T.: $G/H_{n-1} = H_{n-1}K_{m-1}/H_{n-1}$

$$\cong K_{m-1}/(H_{n-1} \cap K_{m-1}).$$

Similarly,

$$G/K_{m-1} \cong H_{n-1}/(H_{n-1} \cap K_{m-1}).$$

So (C) and (D) are isomorphic, and thus so are $(H_i)_{i=0}^{n}$ and $(K_j)_{j=0}^{m}$. ∎