Day 12 June 17, 2024

<u>Recall</u>: The first isomorphism theorem says

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\phi"'] & \psi(G) \leq H \\ G/\mathrm{Ker}\,\psi \arrow[ur, dotted, "\exists!\,\eta"'] & \end{tikzcd}$$

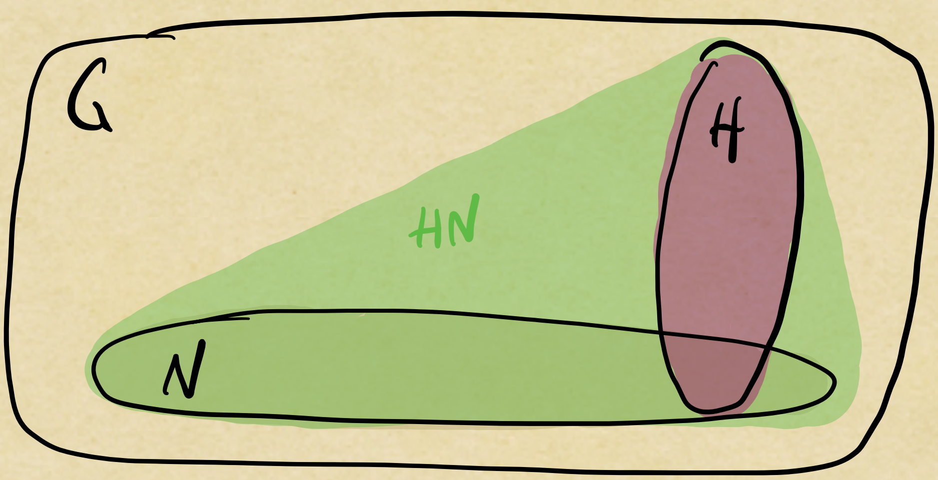
Our next isomorphism theorem addresses the question of what happens to $H \leq G$ when we pass from G to G/N.

<u>Ex</u>. $N = SL_2(\mathbb{R}) \trianglelefteq GL_2(\mathbb{R}) = G$.

$H = O(2) = \{A \in GL_2(\mathbb{R}) \mid A^T A = I\} \leq G$.

Notice: $H \not\subseteq N$ and $N \not\subseteq H$.

So what happens to H in G/N?



$$G \longrightarrow G/N$$
$$g \longmapsto gN$$

Natural guess: $H / H \cap N$.

Concerns: ① Is $H \cap N$ definitely normal in H?

② This is not naturally a subset of G/N.

Thm. [Second Isomorphism Theorem]

Let $H, N \leq G$, with $N \trianglelefteq G$. Then

1. $HN \leq G$ is a subgroup;
2. $H \cap N \trianglelefteq H$ is a normal subgroup of H;
3. the quotient groups

$$HN/N \quad \text{and} \quad H/H\cap N$$

are isomorphic to one another.

(Proof.) 1 and 2 are exercises.

3 Define
$$\phi : H \longrightarrow HN/N$$
$$h \longmapsto hN.$$

Any coset of N in HN has the form
$$(hn)N = h(nN) = hN,$$
so ϕ is surjective. It's also a homom:
$$\phi(h_1h_2) = (h_1h_2)N$$
$$= (h_1N)(h_2N) = \phi(h_1)\,\phi(h_2).$$

By F.I.T.,
$$H/\operatorname{Ker}\phi \cong \phi(H) = HN/N.$$

But $\operatorname{Ker}\phi = \{h \in H \mid hN = N\} = H \cap N$. ∎

Ex. $N = SL_2(\mathbb{R}) \leq GL_2(\mathbb{R}) = G$. $H = O(2)$.
$$HN = \{A \in GL_2(\mathbb{R}) \mid \det A = \pm 1\}$$
$$H \cap N = SO(2).$$

Check: $HN/N \cong \mathbb{Z}_2 \cong H/H\cap N$

Thm [The correspondence theorem]

Let $N \trianglelefteq G$. Then

① there is a bijection

$$\left\{\begin{matrix}\text{subgroups of } G \\ \text{which contain } N\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{subgroups} \\ \text{of } G/N\end{matrix}\right\},$$

given by $H \longleftrightarrow H/N$;

② under this bijection, normal subgroups are taken to normal subgroups.

(Proof.) Since $N = H \cap N$, N is a normal subgroup of H, so H/N is a well-defined quotient group.

Check: H/N is a subgroup of G/N.

To show that $H \mapsto H/N$ is a bijection, we'll construct an inverse. i.e., for every $K \leq G/N$, we'll build $H_K \leq G$ which contains N and check that

$$H_{H/N} = H \quad \& \quad H_K/N = K.$$

For $K \leq G/N$, we define

$$H_K := \{ g \in G \mid gN \in K \}.$$

Since $H_K = \phi^{-1}(K)$, where ϕ is the canonical quotient homom., H_K is a subgroup of G.

For every $n \in N$, $\phi(n) = nN = N \in K$, Since $N \in G/N$ is the identity element.

$\therefore n \in \Phi^{-1}(K)$. So $N \subseteq H_K$.

Exercise: Check that $H_{H/N} = H$ & $H_K/N = K$.

② Let $N \trianglelefteq H \trianglelefteq G$. Consider the homom.

$$G/N \longrightarrow G/H$$
$$gN \longmapsto gH.$$

This map is surjective and its kernel is

$$\{gN \mid gH = H\} = \{gN \mid g \in H\} = H/N.$$

Since H/N is the kernel of a homom., it's normal.

Let $K \trianglelefteq G/N$. Consider

$$G \xrightarrow{\Phi} G/N \longrightarrow (G/N)/K,$$
$$g \longmapsto gN \longmapsto (gN)K.$$

The kernel of this homom is

$$\{g \in G \mid (gN)K = K\} = \{g \in G \mid gN \in K\}$$
$$= \Phi^{-1}(K) = H_K.$$

So H_K is the kernel of a homom, and thus is normal. ∎

Thm [The third isomorphism theorem]
If $N \trianglelefteq H \trianglelefteq G$ and $N \trianglelefteq G$, then

$$G/H \cong (G/N)/(H/N).$$