Day 11 June 13, 2024

Recall: A map $\phi : G \to H$ is a homomorphism if

$$\begin{tikzcd} G \times G \arrow[r, "\circ_G"] \arrow[d, "\phi \times \phi"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "\phi"] \\ H \times H \arrow[r, "\circ_H"] & H \end{tikzcd}$$

__Prop.__ If $\phi : G \to H$ is a homomorphism, then

1. $\phi(e_G) = e_H$;
2. $\phi(g^{-1}) = (\phi(g))^{-1}$, $\forall g \in G$;
3. if $K \leq G$, then $\phi(K) \leq H$;
4. if $K \leq H$, then $\phi^{-1}(K) \leq G$;
5. if $K \trianglelefteq H$, then $\phi^{-1}(K) \trianglelefteq G$.

(Proof of 4 + 5.)

§ $K \leq H$. We'll apply the subgroup criteria to $\phi^{-1}(K) \subseteq G$.

- Nonempty: $\phi(e_G) = e_H \in K \Rightarrow e_G \in \phi^{-1}(K)$.
- Pick $g_1, g_2 \in \phi^{-1}(K)$. Then $\phi(g_1), \phi(g_2) \in K$.

  So
$$\phi(g_1 g_2^{-1}) = \phi(g_1)\phi(g_2^{-1}) = \phi(g_1)(\phi(g_2))^{-1} \in K,$$
  and thus $g_1 g_2^{-1} \in \phi^{-1}(K)$.

Further § $K \trianglelefteq H$. i.e., $hKh^{-1} = K$, $\forall h \in H$.

We NTS $g\,\phi^{-1}(K)\,g^{-1} \subseteq \phi^{-1}(K)$, $\forall g \in G$.

Pick $g' \in \phi^{-1}(K)$. Then
$$\phi(g g' g^{-1}) = \phi(g)\phi(g')\phi(g^{-1}) \in \phi(g) K (\phi(g))^{-1} = K,$$
so $g g' g^{-1} \in \phi^{-1}(K)$. So $g\,\phi^{-1}(K)\,g^{-1} = \phi^{-1}(K)$. $\blacksquare$

Def. Given a homomorphism $\phi: G \to H$, the **kernel** of ϕ is the subset $\ker\phi := \phi^{-1}(e_H)$.

Cor. The kernel of a homomorphism is a normal subgroup.

(Proof.) Given $\phi: G \to H$, $\ker\phi = \phi^{-1}(\{e_H\})$ is the preimage of a normal subgroup of H.

Apply property (5). ∎

Ex. ① $\phi: \mathbb{Z} \to \mathbb{Z}_n$, $k \mapsto k \pmod n$ has $\ker\phi = n\mathbb{Z} \trianglelefteq \mathbb{Z}$.

② $\phi: GL_n(\mathbb{R}) \to \mathbb{R}^*$, $A \mapsto \det A$ has $\ker\phi = SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R})$

③ $\phi: \mathbb{R} \to S^1 \leq \mathbb{C}^*$, $t \mapsto e^{2\pi i t}$ has $\ker\phi = \mathbb{Z} \trianglelefteq \mathbb{R}$

Prop. A homomorphism is injective iff its kernel is trivial.

(Proof.) Let $\phi: G \to H$ be a homom.

($\Rightarrow$) ϕ is injective. Then

$$\ker\phi = \{g \in G \mid \phi(g) = e_H\}.$$

injectivity

But $\phi(g) = e_H$ iff $\phi(g) = \phi(e_G)$ iff $g = e_G$.

So $\ker\phi = \{e_G\}$.

($\Leftarrow$) $\ker\phi = \{e_G\}$. Pick $g_1, g_2 \in G$ s.t. $\phi(g_1) = \phi(g_2)$.

Then $\phi(g_1 g_2^{-1}) = \phi(g_1)(\phi(g_2))^{-1} = e_H$, so $g_1 g_2^{-1} \in \ker\phi$. $\therefore g_1 g_2^{-1} = e_G$, so $g_1 = g_2$. ∎

Ex. Suppose $p, n \in \mathbb{N}$ have $\gcd(p,n) = 1$, with p prime.
Then the only homom. $\mathbb{Z}_p \to \mathbb{Z}_n$ is $\phi(k) := 0$.

If $\phi : \mathbb{Z}_p \to \mathbb{Z}_n$ is any homom., then $\operatorname{Ker}\phi \trianglelefteq \mathbb{Z}_p$, so $\operatorname{Ker}\phi = \{0\}$ or $\ker\phi = \mathbb{Z}_p$.

If $\ker\phi = \{0\}$, then ϕ is injective, so $\phi(\mathbb{Z}_p) \leq \mathbb{Z}_n$ of order p.

Since $p \nmid n$, this is impossible, so $\ker\phi = \mathbb{Z}_p$.

The first isomorphism theorem

normal subgroups = "subgroups we can crush to be the identity"

Kernels = "subgroups crushed to the identity by homomorphisms"

Now: normal subgroups = kernels

Def. Given $N \trianglelefteq G$, the **canonical homomorphism** $\phi : G \to G/N$ is defined by $\phi(g) := gN$.

Thm. [First Isomorphism Theorem] Let $\psi : G \to H$ be a homomorphism and let $N = \operatorname{Ker}\psi$. Then there is a unique isomorphism $\eta : G/N \to \psi(G)$ such that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & H \\
& G/N \arrow[ur, "\eta"'] &
\end{tikzcd}$$

(Proof.) Given $\psi: G \to H$ and $N = \operatorname{Ker}\psi$, we define $\eta : G/N \longrightarrow \psi(G)$

$$gN \longmapsto \psi(g).$$

NTS η is well-defined.

§ $g_1N = g_2N$. Then $g_1^{-1}g_2 \in N$, so

$$e_H = \psi(g_1^{-1}g_2) = (\psi(g_1))^{-1}\psi(g_2),$$

and therefore $\psi(g_1) = \psi(g_2)$.

So $\eta(g_1N) = \eta(g_2N)$. ✓

Homom: $\eta((g_1N)(g_2N)) = \eta((g_1g_2)N)$

$= \psi(g_1g_2) = \psi(g_1)\psi(g_2)$

$= \eta(g_1N)\,\eta(g_2N)$. ✓

Inj: $\eta(g_1N) = \eta(g_2N) \Rightarrow \psi(g_1) = \psi(g_2)$

$\Rightarrow \psi(g_1^{-1}g_2) = e_H$

$\Rightarrow g_1^{-1}g_2 \in N$

$\Rightarrow g_1N = g_2N$. ✓

Surj: Every elt. of $\psi(G)$ has the form $\psi(g)$. ✓

$\blacksquare$

Ex. $\psi : \mathbb{R} \longrightarrow SO_2(\mathbb{R})$

$$t \longmapsto \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$$

$\operatorname{Ker}\psi = \mathbb{Z}$

F.I.T. : $\eta : \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \psi(\mathbb{R})$

Isomorphism theorems tell us about the structure of G/N.

What if we have $N \trianglelefteq G$ and $H \leq G$ with neither $N \subseteq H$ nor $H \subseteq N$?

$$
\begin{tikzcd}[arrows=dash]
 & G \arrow[dl] \arrow[dr] & \\
N \arrow[dr] & & H \arrow[dl] \\
 & N \cap H &
\end{tikzcd}
$$

What happens to H when we pass to G/N?