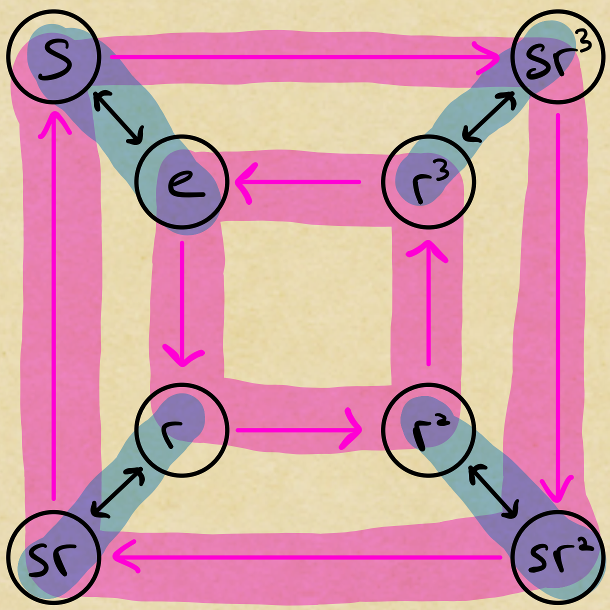
Yesterday: Building up / breaking down groups via direct products.

Today: A different way of decomposing groups.

Consider $G = D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle$



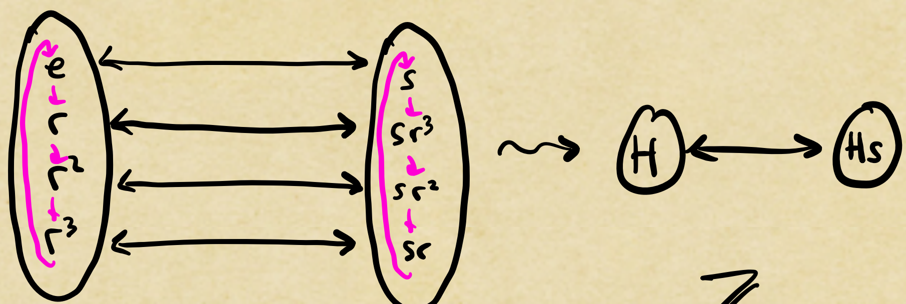
$H = \langle r \rangle$, $K = \langle s \rangle$

$\longleftrightarrow$ = left mult. by s

$\longrightarrow$ = left mult. by r

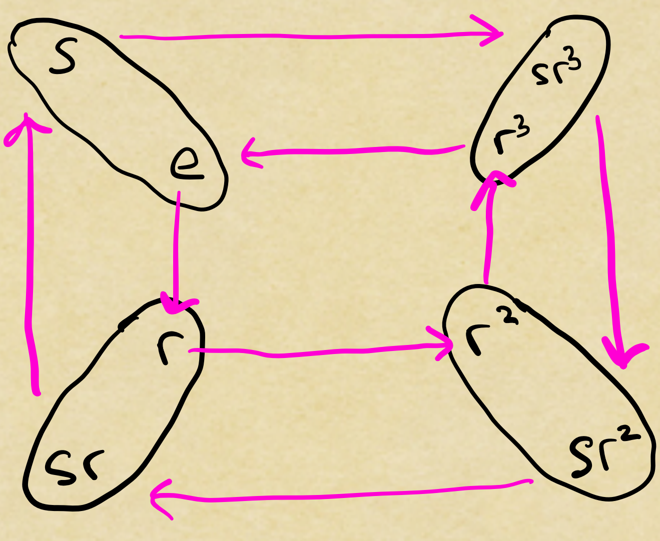
We can find the right cosets of H & K in D_4.

Crush H:



$\mathbb{Z}_2$

Crush K:



The arrows don't allow this!

If we want to crush out $H \leq G$, notice that every $g \in G$ gives a sequence of arrows

$$e \rightsquigarrow g.$$

In the crushed diagram this gives a sequence of arrows

$$H \rightsquigarrow Hg.$$

In fact, $\forall\, h \in H$, we have $h \rightsquigarrow gh$ in the original diagram. In the crushed diagram, this becomes

$$H \rightsquigarrow H(gh).$$

We need $Hg = H(gh)$, $\forall\, h \in H$.

i.e., we need $gh \in Hg$, $\forall\, h \in H$.
equivalently, we need $ghg^{-1} \in H$, $\forall\, h \in H$.

i.e., we want $gHg^{-1} \subseteq H$.

Def. A subgroup $N \leq G$ is called **normal** if $gNg^{-1} \subseteq N$, for every $g \in G$.

Thm. Let $N \leq G$ be a subgroup of a group G. Then TFAE:

1. $gNg^{-1} \subseteq N$, $\forall\, g \in G$;
2. $gNg^{-1} = N$, $\forall\, g \in G$;
3. $gN = Ng$, $\forall\, g \in G$.

Ex ① Every subgroup of an abelian group is normal.

② The subgroup $H = \langle r \rangle \leq D_4$ is normal, while $K = \langle s \rangle \leq D_4$ is not.

③ The subgroup $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$ is normal, by determinant properties:

$A \in GL_n(\mathbb{R})$, $B \in SL_n(\mathbb{R})$

$$\begin{aligned} \det(ABA^{-1}) &= \det(A) \cdot \det(B) \cdot \det(A^{-1}) \\ &= \det(A) \cdot \det(A^{-1}) \\ &= \det(AA^{-1}) = 1. \end{aligned}$$

Thm. Let $N \trianglelefteq G$ be a normal subgroup of G. Then the collection of cosets of N forms a group of order $[G:N]$.

Group operation: $(g_1N)(g_2N) := (g_1g_2)N$.

We call the group of cosets of N in G the quotient group of G by N, denoted G/N.

Ex. ① Every subgroup $n\mathbb{Z} \leq \mathbb{Z}$ is normal, and the quotient group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$ via

$$\begin{aligned} \Phi : \mathbb{Z}_n &\to \mathbb{Z}/n\mathbb{Z} \\ k &\mapsto k + n\mathbb{Z}. \end{aligned}$$

(2) In $D_n = \langle r, s \mid r^n = s^2 = (rs)^2 = 1 \rangle$, the subgroup $R_n = \langle r \rangle$ is normal.
The quotient D_n / R_n has order

$$[D_n : R_n] = \frac{|D_n|}{|R_n|} = \frac{2n}{n} = 2,$$

and thus is isomorphic to $\mathbb{Z}_2$.

__Remark.__ In general, it is not the case that G is isomorphic to $(G/N) \times N$.

$\mathbb{Z} \not\cong (\mathbb{Z}/n\mathbb{Z}) \times n\mathbb{Z} \cong \mathbb{Z}_n \times \mathbb{Z}$

__Def.__ A group is simple if it has no nontrivial, proper, normal subgroups.

__Ex.__ For p prime, $\mathbb{Z}_p$ is simple.

__Thm.__ For $n \geq 5$, the group A_n is simple.

Exercise: Show that A_4 is not simple.

__Rmk.__ As of 2004, there's a complete classification of finite simple groups.

Properties of homomorphisms

Recall: $\Phi : (G, \circ_G) \to (H, \circ_H)$ is a homomorphism

if
$$\begin{tikzcd} G \times G \arrow[r, "\circ_G"] \arrow[d, "\phi \times \phi"'] & G \arrow[d, "\phi"] \\ H \times H \arrow[r, "\circ_H"] & H \end{tikzcd}$$
commutes.

Ex. ① $\forall\ n \geq 1$, $\Phi : \mathbb{Z} \to \mathbb{Z}_n$

$k \mapsto k \pmod n$

② $\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$

③ $\Phi : \mathbb{R} \to S^1 \subset \mathbb{C}$

$t \mapsto \exp(2\pi i t)$.

Note: Only one isomorphism among these examples.

$GL_1(\mathbb{R}) \to \mathbb{R}^*$.