Goal: Build new groups from old, and also reverse this process.

External direct products

Def. The (external) direct product $(G \times H, \circ_{G\times H})$ of the groups $(G, \circ_G)$ & $(H, \circ_H)$ consists of the set

$$G \times H := \{(g,h) \mid g \in G,\ h \in H\}$$

and the operation

$$(g_1, h_1) \circ_{G\times H} (g_2, h_2) := (g_1 \circ_G g_2,\ h_1 \circ_H h_2).$$

Prop. The external direct product is a group.

Ex. Group table for $\mathbb{Z}_2 \times \mathbb{Z}_2$:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

Note: $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4$.

Rmk. Iterating gives a def'n for $G_1 \times G_2 \times \cdots \times G_n$.
Need to check: $(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$.

Ex. For $n \geq 1$, $\mathbb{Z}^n := \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}$ is the free abelian group of rank n.

Ex. Is $\mathbb{Z}_2 \times \mathbb{Z}_3$ cyclic?

$\langle (1,1) \rangle = \{(1,1), (0,2), (1,0), (0,1), (1,2), (0,0)\}$ ✓

$$\begin{array}{cccccc} (0,0), & (1,0), & (0,1), & (1,1), & (0,2), & (1,2) \\ 1 & 2 & 3 & 6 & 3 & 6 \end{array}$$

So $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.

Thm. Let $G = G_1 \times \cdots \times G_n$, where each $G_1, \ldots, G_n$ is a group. Pick elements $g_i \in G_i$, for $1 \leq i \leq n$, and let r_i be the order of g_i in G_i. Then $(g_1, \ldots, g_n) \in G$ has order $\operatorname{lcm}(r_1, \ldots, r_n)$.

(Proof.) Note that $(g_1, \ldots, g_n)^m = e_G$ if and only if

$$(g_1^m, \ldots, g_n^m) = (e_{G_1}, \ldots, e_{G_n}).$$

Since $g_i^m = e_{G_i} \iff m$ is a multiple of r_i,

$(g_1, \ldots, g_n)^m = e_G$ iff m is a multiple of $r_1, r_2, \ldots, r_n$.

The smallest such m is $\operatorname{lcm}(r_1, \ldots, r_n)$. ∎

Thm. Let $n_1, \ldots, n_k$ be positive integers. Then

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \cong \mathbb{Z}_{n_1 n_2 \cdots n_k}$$

iff $\gcd(n_i, n_j) = 1$, $\forall\ 1 \leq i \neq j \leq n$.

(Proof.)

$\S$ $\gcd(n_i, n_j) = 1$, $\forall$ $1 \le i \ne j \le n$.

Then $\operatorname{lcm}(n_1, \ldots, n_k) = n_1 n_2 \cdots n_k$, so $(1, 1, \ldots, 1) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ has order $n_1 n_2 \cdots n_k$ and we're done.

OTOH, $\S$ $\gcd(n_i, n_j) > 1$ for some $i \ne j$. Then $\operatorname{lcm}(n_1, \ldots, n_k) < n_1 n_2 \cdots n_k$. But for any $(a_1, \ldots, a_k) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$,

"exponent"

$$\begin{aligned}
\operatorname{lcm}(n_1, \ldots, n_k) \cdot (a_1, \ldots, a_k) &= (\operatorname{lcm}(n_1, \ldots, n_k) \cdot a_1, \ldots, \operatorname{lcm}(n_1, \ldots, n_k) \cdot a_k) \\
&= (0, \ldots, 0).
\end{aligned}$$

So $(a_1, \ldots, a_k)$ has order at most

$$\operatorname{lcm}(n_1, \ldots, n_k) < n_1 n_2 \cdots n_k.$$

In particular, there is no element of order $n_1 n_2 \cdots n_k$. $\blacksquare$

Cor. For any positive integer m,

$$\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}},$$

where $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime factorization of m.

Important! The primes $p_1, \ldots, p_k$ are distinct.

Internal direct products

Def. Let $H, K \leq G$ be subgroups such that

① $H \cap K = \{e\}$;

② $hk = kh$, $\forall\ h \in H$ and $k \in K$.

Then the internal direct product of H & K is

$$HK := \{hk \mid h \in H, k \in K\} \subseteq G.$$

Prop. When defined, the internal direct product is a subgroup of G.

(Proof.)

- $e \in H \cap K \to e = ee \in HK \to HK \neq \phi$
- Pick $h_1k_1, h_2k_2 \in HK$. Then

$$\begin{aligned}(h_1k_1)(h_2k_2)^{-1} &= (h_1k_1)(k_2^{-1}h_2^{-1}) \\ &= h_1(k_1k_2^{-1})h_2^{-1} \quad \text{cond. ②} \\ &= (h_1h_2^{-1})(k_1k_2^{-1}) \in HK\end{aligned}$$

∎

Ex. ① $U(8) = \{1, 3, 5, 7\}$. $H = \{1, 3\}$ & $K = \{1, 5\}$

$H \cap K = \{1\}$.

$$\begin{aligned}HK &= \{1\cdot 1, 1\cdot 5, 3\cdot 1, 3\cdot 5\} \\ &= \{1, 5, 3, 7\} = U(8).\end{aligned}$$

② $U(10) = \{1, 3, 7, 9\}$ is not the internal direct product of any proper subgroups. This is b/c there's only one subgroup of order 2.

③ D_3 is not the internal direct product of any proper subgroups

D_6 is a nontrivial internal direct product.

Thm. Let $H, K \leq G$ be subgroups s.t. the internal direct product $HK \leq G$ is defined. Then $HK \cong H \times K$.

(Proof.) Consider the map

$$\phi : H \times K \longrightarrow HK$$
$$(h, k) \longmapsto hk.$$

Immediately surjective.

Also injective: $\phi(h_1, k_1) = \phi(h_2, k_2)$

$$\longrightarrow \quad h_1 k_1 = h_2 k_2$$
$$\longrightarrow \quad h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K$$

$\therefore h_2^{-1} h_1 = k_2 k_1^{-1} = e$

$\rightarrow h_1 = h_2$ & $k_1 = k_2$. So $(h_1, k_1) = (h_2, k_2)$.

Homomorphism:

$$\begin{aligned}\phi((h_1, k_1) \cdot (h_2, k_2)) &= \phi(h_1 h_2, k_1 k_2) \\ &= (h_1 h_2)(k_1 k_2) \\ &= (h_1 k_1)(h_2 k_2) \\ &= \phi(h_1, k_1) \cdot \phi(h_2, k_2).\end{aligned}$$

∎