Subgroups of S_n

Transpositions & the alternating subgroup

A transposition is a cycle of length 2.

Recall: Any $\sigma \in S_n$ can be written as a product of disjoint cycles $\sigma_1 \sigma_2 \cdots \sigma_m$.

Thm For $n \geq 2$, any $\sigma \in S_n$ can be written as a product of transpositions.

(Proof.) Given a cycle $(a_1 a_2 \cdots a_k)$, we have

$$(a_1 a_2 \cdots a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_3)(a_1 a_2).$$

For arbitrary $\sigma \in S_n$, first express as a product of cycles, then express cycles as products of transpositions. $\blacksquare$

Factorization into transpositions is not unique, but the parity of the number of transpositions is unique.

Thm. If $\sigma_1 \sigma_2 \cdots \sigma_m = \tau_1 \tau_2 \cdots \tau_n$, with each σ_i & τ_j a transposition, then m & n are either both odd or both even.

(Proof.) Lemma. If (1) is written as the product of r transpositions, then r is even.

(Pf.) Exercise. $\square$

Now left multiply each side of

$$\sigma_1 \sigma_2 \cdots \sigma_m = \tau_1 \tau_2 \cdots \tau_n$$

by σ_1 to obtain

$$\sigma_2 \cdots \sigma_m = \sigma_1 \tau_1 \tau_2 \cdots \tau_n,$$

since $\sigma_1^2 = (1)$. Repeat for $\sigma_2, \dots, \sigma_m$ to obtain

$$(1) = \sigma_m \cdots \sigma_2 \sigma_1 \tau_1 \tau_2 \cdots \tau_n.$$

By lemma, $m+n$ is even $\rightarrow$ $m \equiv n \pmod 2$. $\blacksquare$

Call $\sigma \in S_n$ even if it is a product of an even number of transpositions, odd otherwise.

<u>Thm</u>. The collection of even permutations forms a subgroup of S_n.

(Proof.) (1) is even $\rightarrow$ non-empty.

Say σ, τ are even. Write

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2m}, \quad \tau = \tau_1 \tau_2 \cdots \tau_{2k}$$

Then

$$\sigma \tau^{-1} = \sigma_1 \sigma_2 \cdots \sigma_{2m} \tau_{2k} \cdots \tau_2 \tau_1$$

$\rightarrow$ $2m + 2k$ is even, so $\sigma\tau^{-1}$ is even.

By subgroup criteria, we have a subgroup. $\boxed{M}$

<u>Notation</u>. $A_n = \{\text{even permutations}\} \subseteq S_n$

$= \text{alternating group on } n \text{ letters}$

<u>Motion subgroups</u>

For any $n \geq 3$, the n^{th} dihedral group D_n is the symmetry group of a regular n-gon. A regular n-gon has n vertices, permuted by the elements of $D_n \to D_n \subseteq S_n$.

$$\left(\begin{matrix}n \text{ choices for} \\ \text{vertex } 1\end{matrix}\right) \cdot \left(\begin{matrix}2 \text{ choices for} \\ \text{vertex } 2\end{matrix}\right) = 2n \text{ symmetries.}$$

The dihedral group is neither cyclic nor abelian $(n \geq 3)$.

Not cyclic $\Rightarrow$ Need at least two generators.

$r =$ Rotation by $\frac{2\pi}{n}$ radians

$s =$ reflection . ..

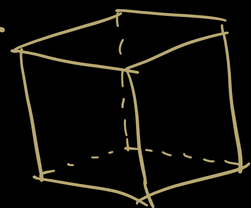
Note that $r^n = (1)$, $s^2 = (1)$. Also, $(sr)^2 = (1)$.

In fact, D_n admits the presentation

$$D_n = \langle r, s \mid r^n = s^2 = (sr)^2 = (1) \rangle.$$

We can play a similar game for the symmetries of Platonic solids.

<u>Cube</u>



Of which S_n is this a subgroup?

Motion subgroup can be realized as a subgroup of S_8, S_{12}, S_6.

$\uparrow$ V $\quad \uparrow$ E $\quad \uparrow$ F

In fact, by tracking the diagonals instead, we can realize the group of rigid motions as S_4.

<u>Tetrahedron</u>: $A_4 \subsetneq S_4$

<u>Octahedron</u>: S_4

<u>Dodecahedron & icosahedron</u>: A_5

Cosets

<u>Def'n.</u> Let $H \leq G$ be a subgroup of a group G. The left coset of H with representative $g \in G$ is the set

$$gH := \{gh \mid h \in H\}$$

and the right coset of H with representative $g \in G$ is

$$Hg := \{hg \mid h \in H\}.$$

Ex.

① Recall: $n\mathbb{Z} \subseteq \mathbb{Z}$ is a subgroup, not a coset.

Cosets of $n\mathbb{Z}$ look like $m + n\mathbb{Z}$.

e.g., $m + 3\mathbb{Z} = \{\dots, m-6, m-3, m, m+3, m+6, \dots\}$.

$3\mathbb{Z} \subset \mathbb{Z}$ has three cosets:

$$0 + 3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \dots\}$$
$$1 + 3\mathbb{Z} = \{\dots, -5, -2, 1, 4, 7, \dots\}$$
$$2 + 3\mathbb{Z} = \{\dots, -4, -1, 2, 5, 8, \dots\}.$$

$$m_1 + 3\mathbb{Z} = m_2 + 3\mathbb{Z} \iff m_1 \equiv m_2 \pmod 3$$

② $H = \{(1), (12)\} \subset S_3$.

Left cosets:

$$(1)H = \{(1), (12)\}$$
$$(12)H = \{(12), (1)\}$$
$$(13)H = \{(13), (13)(12)\} = \{(13), (123)\}$$
$$(23)H = \{(23), (23)(12)\} = \{(23), (132)\}$$
$$(123)H = \{(123), (123)(12)\} = \{(123), (13)\}$$
$$(132)H = \{(132), (132)(12)\} = \{(132), (23)\}$$

Exercise: Right cosets $\neq$ left cosets in this example