$\boxed{\text{Midterm on Friday covers through group actions.}}$

The Sylow theorems

Recall: We know how to classify finite abelian groups, and would like to classify all finite groups.

Sylow theorems will give us info about the subgroups of G based $|G|$.

Idea: We already have $(H \leq G \Rightarrow |H| \mid |G|)$. We'll get partial converses.

$|A_4| = 12$
No subgroup of order 6.

We already have Cauchy's theorem in abelian case:

G abelian & $p \mid |G| \Rightarrow \exists g \in G$ s.t. $|g| = p$.

prime

Thm. (First Sylow Theorem) If p is prime and $p^r \mid |G|$, then $\exists$ subgroup $H \leq G$ s.t. $|H| = p^r$.

(Proof.) We will induct on $|G|$.

Base case: $|G| = p$. Then p^1 is the only power of p dividing $|G|$. $H = G$ is our subgroup.

Inductive step will need the class equation:

$$|G| = |Z(G)| + \sum_i [G : C_G(g_i)].$$

the g_i represent distinct conjugacy classes

- if $p \nmid [G:C_G(g_i)]$, then $p^r \mid |C_G(g_i)|$,

  b/c $\displaystyle |C_G(g_i)| = \frac{|G|}{[G:C_G(g_i)]}$. ← $p^r \mid |G|$ ← $p \nmid [G:C_G(g_i)]$

  Note that $|C_G(g_i)| < |G|$. If $C_G(g_i) = G$, then $g_i \in Z(G)$ is its own (trivial) conjugacy class. So the inductive hypothesis gives

  $$H \leq C_G(g_i) \text{ with } |H| = p^r.$$

  i.e., $H \leq C_G(g_i) \leq G$ w/ $|H| = p^r$.

- OTOH, if $p \mid [G:C_G(g_i)]$, $\forall i$, then p divides $|Z(G)|$, according to the class equation. (And $|Z(G)| \neq 0$.) So $Z(G)$ is an abelian group with $p \mid |Z(G)| \Rightarrow \exists\, g \in Z(G)$ s.t. $|g| = p$.

  Let $N = \langle g \rangle \leq G$. Since $g \in Z(G)$, N is normal.

  $$|G/N| = \frac{|G|}{|N|} < |G| \text{ \& } p^{r-1} \mid |G/N|.$$

  Inductive hypothesis $\Rightarrow \exists\, H' \leq G/N$ with $|H'| = p^{r-1}$.

  By correspondence thm, $\exists\, H \leq G$ with $N \leq H$ s.t. $H/N = H'$.

Check: $|H| = p^r$.

Def A Sylow p-subgroup of G is a subgroup of order p^n, where $|G| = p^n m$, $p \nmid m$.

(p-group / subgroup $\Rightarrow |H| = p^{\wedge}$, Sylow $\Rightarrow$ maximal)

Interlude: G acting on its subgroups

Throughout, G is a finite group.

Let $\mathcal{G}$ = collection of all subgroups of G. Then $G \curvearrowright \mathcal{G}$.

$$\begin{aligned} G &\longrightarrow S_{\mathcal{G}} \\ g &\longmapsto (H \mapsto gHg^{-1}). \end{aligned}$$

Def. The normalizer of $H \le G$, denoted $N(H)$, is the stabilizer subgroup of $H \in \mathcal{G}$ under the action of G on $\mathcal{G}$ by conjugation:

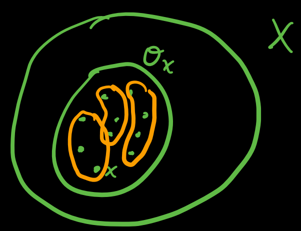
$$N(H) := \{ g \in G \mid gHg^{-1} = H \}.$$

Exercise. $N(H)$ is the largest subgroup of G of which H is a normal subgroup.

Note that whenever we have a group action $G \curvearrowright X$ and a subgroup $H \le G$, we get $H \curvearrowright X$.

Exercise. $\forall x \in X$, $H \le G$ acts on $\mathcal{O}_x$.

i.e., the orbits $\mathcal{O}_y^H = \{ h \cdot y \mid h \in H \}$, $y \in \mathcal{O}_x$, partition $\mathcal{O}_x$.



Prop ① Fix any subgroups $H, K \leq G$, with $H \curvearrowright G$ by conjugation. Then $|\mathcal{O}_K^H| = [H : N(K) \cap H]$.

(Proof.) For any $H \curvearrowright X$ and any $x \in X$, we've seen

$$|\mathcal{O}_x^H| = [H : H_x],$$

where $H_x = \{h \in H \mid h.x = x\}$.

In our case, $X = G$ & $x = K \in G$, so

$$\begin{aligned} H_K &= \{h \in H \mid h.K = K\} \\ &= \{h \in H \mid hKh^{-1} = K\} \\ &= H \cap N(K). \end{aligned}$$

∎

Counting Sylow p-subgroups

Prop ② If $P \leq G$ is a Sylow p-subgroup and $x \in N(P)$ satisfies $|x| = p^k$, for some k, then $x \in P$.

(Proof.) Exercise. Hint: Consider $\langle xP \rangle \leq N(P)/P$. ∎

__Thm__ (Second Sylow Theorem) For any prime p and finite group G, the Sylow p-subgroups of G are pairwise G-conjugate.

i.e., the Sylow p-subgroups of G constitute a single orbit of the action $G \circlearrowright \mathcal{S}$.