Applications of the class equation

Recall: If G is a finite group with r distinct conjugacy classes, rep'd by $g_1, \ldots, g_r$, then

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$

$G \curvearrowright G$
$g.h = ghg^{-1}$

Conjugacy class = orbit with more than one element

$C_G(g_i)$ = centralizer subgroup of g_i $\{g \in G \mid g.g_i = g_i\}$

Thm. Any group of prime-power order has nontrivial center.

(Proof.) Suppose G is finite, with $|G| = p^n$, for some prime p and some integer n.

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)]$$

Lagrange: $[G : C_G(g_i)] = \dfrac{|G|}{|C_G(g_i)|} = p^{k_i}$, for some $k_i \geq 0$.

In fact, $k_i \geq 1$, because $|\mathcal{O}_x| = [G : G_x]$ for any action. If $[G : C_G(g_i)] = p^0 = 1$, then the conjugacy class has size 1. But conjugacy classes always have >1 element.

In particular, $p \mid [G : C_G(g_i)]$.

Rewriting the class equation:

Not zero b/c $e \in Z(G)$

$$|Z(G)| = |G| - \sum_{i=1}^{r} [G : C_G(g_i)].$$

Since RHS is divisible by p, so is $|Z(G)|$. ∎

<u>Cor</u> Any group of order p^2, with p prime, is abelian.

(Proof.) Take G with $|G| = p^2$.

From Thm, $|Z(G)| > 1$. So $|Z(G)| = p$ or $|Z(G)| = p^2$.

If $|Z(G)| = p^2$, then $Z(G) = G$ and we win.

If $|Z(G)| = p$, then $Z(G)$ is cyclic.

Also, $Z(G)$ is always normal in G, so $G/Z(G)$ makes sense, and $|G/Z(G)| = p$. So $G/Z(G)$ is also cyclic.

Let $aZ(G) \in G/Z(G)$ be a generator.

Any $g \in G \rightarrow gZ(G) = (aZ(G))^m = a^m Z(G)$, for some m.

$\therefore a^{-m} g \in Z(G)$.

Take $g_1, g_2 \in G$ and choose m_1, m_2 s.t.

$a^{-m_1} g_1, a^{-m_2} g_2 \in Z(G)$

Write $a^{-m_1} g_1 = x_1 \in Z(G)$ & $a^{-m_2} g_2 = x_2 \in Z(G)$.

Then

$$g_1 g_2 = (a^{m_1} x_1)(a^{m_2} x_2) = a^{m_1} a^{m_2} x_1 x_2 \quad \text{in } Z(G)$$

$$= a^{m_2} a^{m_1} x_2 x_1 = a^{m_2} x_2 a^{m_1} x_1$$

$$= g_2 g_1.$$

So G is abelian. $\blacksquare$

Burnside's counting theorem

Let G be a finite group acting on a set X, and let k be the number of orbits of X. Then

$$k = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

Rmk. In words: the number of orbits is the average number of fixed points.

e.g., if G acts trivially $\Rightarrow X_g = X \Rightarrow$ average size of $X_g = |X|$

$\Rightarrow$ each elt of X is an orbit $\Rightarrow |X|$ orbits

(Proof) Let's rewrite the sum:

$$\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{|G|} \sum_{g \in G} \Big| \overbrace{\{x \in X \mid g.x = x\}}^{\text{fixed point set of } g \in G} \Big|$$

$$= \frac{1}{|G|} \left| \{ (x,g) \in X \times G \mid g.x = x \} \right|$$

$$= \frac{1}{|G|} \sum_{x \in X} \Big| \underbrace{\{ g \in G \mid g.x = x \}}_{\text{stabilizer subgroup of } x \in X.} \Big|$$

$$= \frac{1}{|G|} \sum_{x \in X} |G_x|.$$

Lagrange: $[G : G_x] = \dfrac{|G|}{|G_x|} \Rightarrow |G_x| = \dfrac{|G|}{[G : G_x]}.$

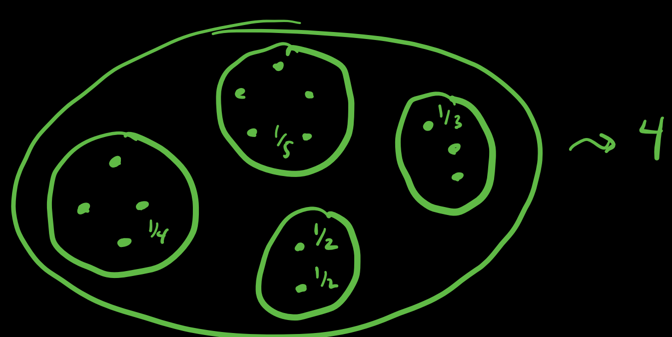
$$\therefore \quad \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{|G|} \sum_{x \in X} |G_x|$$

$$= \frac{1}{\cancel{|G|}} \sum_{x \in X} \frac{\cancel{|G|}}{[G : G_x]}$$

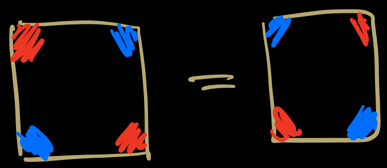
$$= \sum_{x \in X} \frac{1}{[G : G_x]}.$$

But $\forall x \in X, \ |O_x| = [G : G_x].$

So $$\frac{1}{|G|} \sum_{g \in G} |X_g| = \underbrace{\sum_{x \in X} \frac{1}{|O_x|} = \#\text{ of orbits}} \qquad \blacksquare$$



__Ex__ How many ways can we color the corners of a square with two colors?



(apply r)

$$G = D_4 = \left\{ \begin{array}{l} (1),\ (1234), (13)(24),\ (1432), \\ (12)(34),\ (24),\ (14)(23),\ (13) \end{array} \right\}$$

$$X = \left\{ \text{BBBB},\ \text{BBBR},\ \text{BBRB},\ \text{BBRR} \right.$$

$$\left. \right\}$$

= all possible colorings of a fixed square (no symmetries applied yet)

$G \curvearrowright X$: $(12)(34) . \text{BBBR} = \text{BBRB}$

\# of distinct colorings when symmetries are allowed = \# of orbits of $G \curvearrowright X$.

$$X_e = X \Rightarrow |X_e| = 16$$

$$X_r = \{\text{BBBB}, \text{RRRR}\} \Rightarrow |X_r| = 2$$

$$X_{r^2} = \{\text{BBBB}, \text{BRBR}, \text{RBRB}, \text{RRRR}\} \Rightarrow |X_{r^2}| = 4$$

⋮

After computing all 8 fixed point sets, compute their average size.

$$k = \frac{1}{8}(16+2+4+2+4+8+4+8) = \frac{48}{8} = \boxed{6}.$$