Group actions

Idea: We can learn a lot about a group by studying sets on which it produces symmetries.

Def. For any set X & any group G, a (left) action of G on X is a homomorphism $G \to S_X$. Given an action of G on X, we call X a G-set.

Rmk. We can also write actions as maps $\rho : G \to S_X$

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g \cdot x \quad \rho(g)(x)$$

Satisfying (1) $e \cdot x = x, \ \forall x \in X$; $(\text{i.e., } \rho(e_G) = id)$

(2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x), \ \forall g_1, g_2 \in G$ & $x \in X$.

$(\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2))$

"acts on"

Ex ① $S_n \curvearrowright \{1, 2, \ldots, n\}$. The homom. $S_n \to S_n$ is id.

② $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$. $GL_n(\mathbb{R}) \longrightarrow S_{\mathbb{R}^n}$

$$A \longmapsto (\vec{v} \longmapsto A\vec{v})$$

in std basis

③ $D_n \curvearrowright \binom{\text{regular}}{n\text{-gon}}$. $D_n \longrightarrow S_n$ is obtained by labeling the vertices of our n-gon and treating each elt. of D_n as a permutation of these.

④ The most important example

Any group G acts on itself:

$$G \longrightarrow S_G$$

$$g \longmapsto (h \mapsto gh)$$ ← left regular representation

This is fine, but not an action which preserves group structure. Better:

$$G \longrightarrow S_G$$

$$g \longmapsto (h \mapsto ghg^{-1})$$

i.e., G acts on itself by **conjugation**

<u>Def</u>. Let X be a G-set. Call x_1 & $x_2 \in X$ **G-equivalent**, written $x_1 \sim_G x_2$, if

$$\exists\, g \in G \text{ s.t. } g.x_1 = x_2.$$

<u>Ex</u> $S^1 \curvearrowright S^2$

$G = S^1 \subset \mathbb{C}$, $X = S^2 =$ sphere

G-equivalence classes are latitudes, plus the North and South poles

Prop For any G-set X, G-equivalence is an equivalence rel'n.

(Proof.) Exercise. ☒

Def. Let X be any G-set.

① The orbits of X under G are the G-equivalence classes. $\forall x \in X$, $\mathcal{O}_x$ = the orbit containing x.

② The fixed point set of a given $g \in G$ is

$$X_g := \{x \in X \mid g.x = x\}.$$

③ The stabilizer subgroup (or isotropy subgroup) of a given $x \in X$ is $G_x := \{g \in G \mid g.x = x\}$.

Prop. The stabilizer subgroup is a subgroup of G.

(Proof.) Exercise. ☒

Def. In the action of G on itself by conjugation,

① the conjugacy classes of G are the nontrivial orbits — i.e., the orbits which contain more than one element;

② two elements are conjugate if they are in the same conjugacy class;

③ the centralizer subgroup of $x \in G$ is the stabilizer subgroup: $C_G(x) = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\}$.

Thm. Let G be a finite group, X any finite G-set. For any $x \in X$, $|\mathcal{O}_x| = [G : G_x]$.

(Proof.) Recall: $[G:G_x] = |\mathcal{L}_{G_x}|$

the collection of left cosets of G_x in G

We'll build a bijection

$$\phi : \mathcal{O}_x \longrightarrow \mathcal{L}_{G_x}.$$

Define $\phi(y) := g\,G_x$, where $g.x = y$.

This is well-defined: given $y \in \mathcal{O}_x$, & $g_1.x = y = g_2.x$.

Then $(g_1^{-1}g_2).x = x$. So $g_1^{-1}g_2 \in G_x$.

$\therefore\ g_1 G_x = g_2 G_x$. ✓

Exercise: Check that ϕ is a bijection. ▨

Thm (The class equation)

Suppose G is a finite group with r distinct conjugacy classes, and let $g_1, \dots, g_r$ be representatives of these. Then

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$

(Proof.) Given any $G \curvearrowright X$, with X finite, we have

$$|X| = |X_G| + \sum_{i=1}^{r} |\mathcal{O}_{x_i}|, \qquad (\star)$$

where $X_G = \{x \in X \mid g.x = x, \forall g \in G\}$

and $x_1, \ldots, x_r$ are representatives of the r nontrivial orbits of the action.

Why? Because the orbits partition X. In $(\star)$ we've lumped the trivial orbits together into X_G.

For G acting on itself by conjugation,

$$X_G = Z(G) \quad \& \quad |\mathcal{O}_{g_i}| = [G : G_{g_i}] = [G : C_G(g_i)]$$

for g_i a rep. of a $\underbrace{\text{nontrivial orbit}}_{\text{conjugacy class}}$.

So $(\star)$ becomes

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$

$\blacksquare$