

Group actions

Idea: We can learn a lot about a group by studying sets on which it produces symmetries.

Def. For any set X & any group G , a (left) action of G on X is a homomorphism $G \rightarrow S_X$. Given an action of G on X , we call X a G -set.

Rmk. We can also write actions as maps $\rho: G \rightarrow S_X$

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$

$\rho(g)(x)$

satisfying (1) $e \cdot x = x, \forall x \in X$; (i.e., $\rho(e_G) = \text{id}$)

(2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x), \forall g_1, g_2 \in G, x \in X$.

$$(\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2))$$

"acts on"
↓

Ex ① $S_n \curvearrowright \{1, 2, \dots, n\}$. The homom. $S_n \rightarrow S_n$ is id.

② $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$. $GL_n(\mathbb{R}) \longrightarrow S_{\mathbb{R}^n}$

$$A \longmapsto (\vec{v} \longmapsto A\vec{v})$$

↑ in std basis

③ $D_n \curvearrowright$ (regular n -gon).

$D_n \longrightarrow S_n$ is obtained

by labeling the vertices of our n -gon and treating each elt. of D_n as a permutation of these.

④ The most important example

Any group G acts on itself:

$$G \longrightarrow S_G$$

$$g \longmapsto (h \mapsto gh) \longleftarrow \text{left regular representation}$$

This is fine, but not an action which preserves group structure. Better:

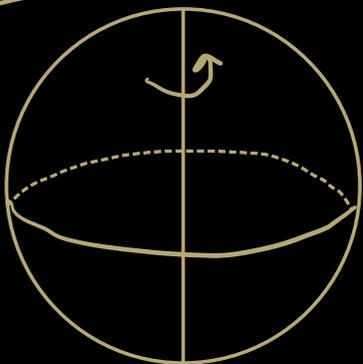
$$G \longrightarrow S_G$$

$$g \longmapsto (h \mapsto ghg^{-1})$$

i.e., G acts on itself by conjugation

Def. Let X be a G -set. Call $x_1, x_2 \in X$ G -equivalent, written $x_1 \sim_G x_2$, if $\exists g \in G$ s.t. $g \cdot x_1 = x_2$.

Ex $S^1 \curvearrowright S^2$



$G = S^1 \subset \mathbb{C}$, $X = S^2 = \text{sphere}$

G -equivalence classes are latitudes, plus the North and South poles

Prop For any G -set X , G -equivalence is an equivalence rel'n.
(Proof.) Exercise. □

Def. Let X be any G -set.

① The **orbits** of X under G are the G -equivalence classes.
 $\forall x \in X$, $\mathcal{O}_x =$ the orbit containing x .

② The **fixed point set** of a given $g \in G$ is
$$X_g := \{x \in X \mid g \cdot x = x\}.$$

③ The **stabilizer subgroup** (or **isotropy subgroup**) of a given $x \in X$ is $G_x := \{g \in G \mid g \cdot x = x\}.$

Prop. The stabilizer subgroup is a subgroup of G .
(Proof.) Exercise. □

Def. In the action of G on itself by conjugation,

① the **conjugacy classes** of G are the nontrivial orbits — i.e., the orbits which contain more than one element;

② two elements are **conjugate** if they are in the same conjugacy class;

③ the **centralizer subgroup** of $x \in G$ is the stabilizer subgroup: $C_G(x) = \{g \in G \mid g x g^{-1} = x\} = \{g \in G \mid g x = x g\}.$

Thm. Let G be a finite group, X any finite G -set.

For any $x \in X$, $|\mathcal{O}_x| = [G : C_x]$.

(Proof.) Recall: $[G : C_x] = |\mathcal{L}_{C_x}|$

↑ the collection of left cosets of C_x in G

We'll build a bijection

$$\phi: \mathcal{O}_x \rightarrow \mathcal{L}_{C_x}.$$

Define $\phi(y) := gC_x$, where $g \cdot x = y$.

This is well-defined; given $y \in \mathcal{O}_x$, $\exists g_1 \cdot x = y = g_2 \cdot x$.

Then $(g_1^{-1}g_2) \cdot x = x$. So $g_1^{-1}g_2 \in C_x$.

$$\therefore g_1 C_x = g_2 C_x. \quad \checkmark$$

Exercise: Check that ϕ is a bijection. □

Thm (The class equation)

Suppose G is a finite group with r distinct conjugacy classes, and let g_1, \dots, g_r be representatives of these. Then

$$|G| = |Z(G)| + \sum_{i=1}^r [G : C_G(g_i)].$$

(Proof.) Given any $G \curvearrowright X$, with X finite, we have

$$|X| = |X_G| + \sum_{i=1}^r |\mathcal{O}_{x_i}|, \quad (\star)$$

where $X_G = \{x \in X \mid g \cdot x = x, \forall g \in G\}$

and x_1, \dots, x_r are representatives of the r nontrivial orbits of the action.

Why? Because the orbits partition X . In (\star) we've lumped the trivial orbits together into X_G .

For G acting on itself by conjugation,

$$X_G = Z(G) \quad \frac{1}{r} |O_{g_i}| = [G : C_G(g_i)] = [G : C_G(g_i)]$$

for g_i a rep. of a nontrivial orbit
" conjugacy class

So (\star) becomes

$$|G| = |Z(G)| + \sum_{i=1}^r [G : C_G(g_i)].$$

\square