Recall: A composition series for G is a finite sequence $(H_i)_{i=0}^{n}$ of nested subgroups with $H_{i-1} \trianglelefteq H_i$ normal & H_i/H_{i-1} simple.

Fact. Any finite group admits a composition series.

Ex ① $\langle 0 \rangle \underset{\mathbb{Z}_2}{\trianglelefteq} \langle 15 \rangle \underset{\mathbb{Z}_3}{\trianglelefteq} \langle 5 \rangle \underset{\mathbb{Z}_5}{\trianglelefteq} \mathbb{Z}_{30}$ isomorphic

& $\langle 0 \rangle \underset{\mathbb{Z}_2}{\trianglelefteq} \langle 15 \rangle \underset{\mathbb{Z}_5}{\trianglelefteq} \langle 3 \rangle \underset{\mathbb{Z}_3}{\trianglelefteq} \mathbb{Z}_{30}$

$$\mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

② For $n \geq 5$, $\{(1)\} \trianglelefteq A_n \trianglelefteq S_n$ is a composition series, b/c A_n is simple & $S_n/A_n \cong \mathbb{Z}_2$.

Thm (Jordan–Hölder) Any two composition series of a fixed G are isomorphic.

(Proof.) Suppose $(H_i)_{i=0}^{n}$ and $(K_j)_{j=0}^{m}$ are composition series for G. We will induct on n.

Base case: $n = 1 \Rightarrow G$ is simple
$\Rightarrow \{e\} \trianglelefteq G$ is the only comp. series.

$\$$ the theorem holds for any comp. series of length $< n$.

Note that if $H_{n-1} = K_{m-1}$, then we're finished b/c

$(H_i)_{i=0}^{n-1}$ & $(K)_{j=0}^{m-1}$ are comp. series for H_{n-1},

hence isomorphic by inductive hypothesis.

If $H_{n-1} \neq K_{m-1}$, we'll construct a pair of comp. series for $H_{n-1} \cap K_{m-1}$, one of which has length $n-1$. I.H. will tell us that they're isomorphic.

(CS 1) $\{e\} = H_0 \cap K_{m-1} \leq H_1 \cap K_{m-1} \leq \cdots \leq H_{n-1} \cap K_{m-1}$

(CS 2) $\{e\} = H_{n-1} \cap K_0 \leq H_{n-1} \cap K_1 \leq \cdots \leq H_{n-1} \cap K_{m-1}$,

with nonproper inclusions removed.

NTS: $(H_i \cap K_{m-1}) / (H_{i-1} \cap K_{m-1})$ is simple,

and something similar for (CS2).

Second isomorphism theorem:

$$\frac{H_i \cap K_{m-1}}{H_{i-1} \cap K_{m-1}} = \frac{H_i \cap K_{m-1}}{H_{i-1} \cap (H_i \cap K_{m-1})} \cong \frac{H_{i-1}(H_i \cap K_{m-1})}{H_{i-1}}$$

ok b/c $H_{i-1} \leq H_i$

This is a normal subgroup of H_i/H_{i-1}. (Exercise)

But! H_i/H_{i-1} is simple.

So $\dfrac{H_i \cap K_{m-1}}{H_{i-1} \cap K_{m-1}} \cong \{e\}$ or H_i/H_{i-1}.

In either case, $\dfrac{H_i \cap K_{m-1}}{H_{i-1} \cap K_{m-1}}$ is simple.

So (CS1) & (CS2) are comp. series for $H_{n-1} \cap K_{m-1}$. By I.H., (CS1) $\cong$ (CS2).

Consider

(CS3) $\{e\} = H_0 \cap K_{m-1} \leq H_1 \cap K_{m-1} \leq \cdots \leq H_{n-1} \cap K_{m-1} \leq H_{n-1}$.

This is a C.S. for H_{n-1}, so it's isomorphic to $(H_i)_{i=0}^{n-1}$. Therefore

(CS4) $\{e\} = H_0 \cap K_{m-1} \leq H_1 \cap K_{m-1} \leq \cdots \leq H_{n-1} \cap K_{m-1} \leq H_{n-1} \leq G$

is isomorphic to $(H_i)_{i=0}^{n}$. Analogously,

(CS5) $\{e\} = H_{n-1} \cap K_0 \leq H_{n-1} \cap K_1 \leq \cdots \leq H_{n-1} \cap K_{m-1} \leq K_{m-1} \leq G$

is isomorphic to $(K_j)_{j=0}^{m}$.

We'll win if we can show that

$$\left\{ \frac{G}{H_{n-1}}, \frac{H_{n-1}}{H_{n-1} \cap K_{m-1}} \right\} = \left\{ \frac{G}{K_{m-1}}, \frac{K_{m-1}}{H_{n-1} \cap K_{m-1}} \right\}.$$

Since $H_{n-1} \neq K_{m-1}$, $H_{n-1} K_{m-1} \leq G$ is normal and properly contains H_{n-1}. But G/H_{n-1} is simple,

so $H_{n-1}K_{m-1} = G$.

Second isom thm:

$$\frac{G}{H_{n-1}} = \frac{H_{n-1}K_{m-1}}{H_{n-1}} \cong \frac{K_{m-1}}{H_{n-1} \cap K_{m-1}}.$$

Analogously, $G/K_{m-1} \cong H_{n-1}/H_{n-1} \cap K_{m-1}$. ∎

Solvability

We call a group G solvable if there is a subnormal series $(H_i)_{i=0}^{n}$ of G s.t. each factor H_i/H_{i-1} is abelian, $1 \leq i \leq n$.

Ex. For $n \leq 4$, S_4 is solvable.

- S_1, S_2 are abelian $\to$ done
- $\{(1)\} \trianglelefteq \langle(123)\rangle \trianglelefteq S_3$
  $\mathbb{Z}_3$ $\mathbb{Z}_2$
- $\{(1)\} \trianglelefteq \{(1),(12)(34),(13)(24),(14)(23)\} \trianglelefteq A_4 \trianglelefteq S_4$
  $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\mathbb{Z}_3$ $\mathbb{Z}_2$

<u>Prop</u> S_n is not solvable, for $n \geq 5$.

(Proof.) Say $(H_i)_{i=0}^m$ is a subnormal series of S_n with H_i/H_{i-1} abelian, for $1 \leq i \leq m$.

Using earlier algorithm (& finiteness of S_n), refine this to a composition series. So we get a composition series for S_n where each factor is abelian. But Jordan-Hölder says

$$\{e\} \trianglelefteq A_n \trianglelefteq S_n$$

is the only CS of S_n, up to isom., and A_n is not abelian. $\blacksquare$