Given $H, N \leq G$, with N normal, what becomes of H in the quotient G/N?

NOT necessarily an internal direct product

<u>Thm</u> (Second isomorphism theorem)

Let $H, N \leq G$ be subgroups, with N normal. Then

1. $HN \subseteq G$ is a subgroup;
2. $H \cap N \trianglelefteq H$ is a normal subgroup of H;
3. the quotient groups

$$HN/N \quad \text{and} \quad H/(H \cap N)$$

are isomorphic.

(Proof.) 1 & 2 exercise

3 Define $\phi : H \longrightarrow HN/N$

$\qquad h \longmapsto hN.$

Surjective: given $h_n N \in HN/N$, $\phi(h) = hN = h_n N$.

homom: $\phi(h_1 h_2) = (h_1 h_2)N = (h_1 N)(h_2 N)$

$\qquad = \phi(h_1)\,\phi(h_2).$

F.I.T.: $H/\ker\phi \cong \phi(H) = HN/N$

Let's compute $\ker\phi$.

$\ker\phi = \{h \in H \mid \phi(h) = N\} = \{h \in H \mid hN = N\}$

$\qquad = \{h \in H \mid h \in N\} = H \cap N.$

So $H/(H \cap N) \cong HN/N$. ∎

Ex. $G = GL_2(\mathbb{R})$. $H = O(2)$. $N = SL_2(\mathbb{R})$.

$$HN = \{AB \mid A \in O(2),\ B \in SL_2(\mathbb{R})\}$$
$$= \{A \in GL_2(\mathbb{R}) \mid \det A = \pm 1\}$$

$$\therefore HN/N = \frac{\{A \in GL_2(\mathbb{R}) \mid \det A = \pm 1\}}{\{A \in GL_2(\mathbb{R}) \mid \det A = 1\}} \cong \mathbb{Z}_2$$

$$H \cap N = O(2) \cap SL_2(\mathbb{R}) = SO_2(\mathbb{R})$$

$$H/(H \cap N) = O(2)/SO(2) \cong \mathbb{Z}_2$$

We can now identify all subgroups of G/N.

Thm (The correspondence theorem)

Let $N \leq G$ be a normal subgroup. Then

1. there is a bijection btwn the subgroups of G which contain N and the subgroups of G/N, given by $H \mapsto H/N$;
2. under this bijection, the normal subgroups of G/N correspond to normal subgroups of G which contain N.

(Proof.)

Step 1: $H \mapsto H/N$ is well-defined

$N \leq H$ and N is normal in G $\Rightarrow$ N is normal in H

Need to check that $H/N \leq G/N$.

nonempty : $N \in H/N$ ✓

$(h_1 N)(h_2 N)^{-1}$: $(h_1 N)(h_2 N)^{-1} = (h_1 N)(h_2^{-1} N)$

$= (h_1 h_2^{-1}) N \in H/N$ ✓

Step ② $H \mapsto H/N$ is bijective

We build an inverse correspondence.

Given $K \leq G/N$, define $H_K := \{ g \in G \mid gN \in K \}$.

H_K is nonempty: $e_G \in H_K$

H_K is subgroup: given $h_1, h_2 \in H_K$, $h_1 N, h_2 N \in K$.

So $(h_1 N)(h_2 N)^{-1} \in K$

$\therefore (h_1 h_2^{-1}) N \in K$

$\therefore h_1 h_2^{-1} \in H_K$ ✓

H_K contains N : given $n \in N$, $nN = N \in K$.

$\therefore n \in H_K$ ✓

So $H_K \leq G$ is a subgroup containing N.

Check: $H_K/N = K$ & $H_{H/N} = H$.

Then $K \mapsto H_K$ is an inverse for $H \mapsto H/N$.

Step ③ If $H \leq G$ is normal, then H/N is normal

Suppose $H \leq G$ is normal and contains N. Consider

$$G/N \longrightarrow G/H$$
$$gN \longmapsto gH$$

well-defined : $g_1 N = g_2 N \to g_1^{-1} g_2 \in N \to g_1^{-1} g_2 \in H$

$\to g_1 H = g_2 H$ ✓

Surjective: any elt of G/H looks like gH, is the image of gN

kernel: $gH = H \leftrightarrow g \in H$

So kernel consists of elts $hN \in G/N$, for $h \in H$.

i.e., kernel is $H/N \leq G/N$.

$\therefore H/N \leq G/N$ is normal

<u>Step ④ If $K \leq G/N$ is normal, then H_K is normal.</u>

Consider $G \longrightarrow G/N \longrightarrow \dfrac{G/N}{K}$

$g \longmapsto gN \longmapsto (gN)K.$

$g \mapsto K \quad \longleftrightarrow \quad gN \in K \longleftrightarrow g \in H_K$

So $H_K \leq G$ is the kernel of this homomorphism.

$\therefore H_K$ is normal. ∎

Thm (Third isomorphism theorem)

Let $N \subseteq H \subseteq G$ be nested normal subgroups.

Then $(G/N)\big/(H/N) \cong G/H$.

(Proof) Recall the homom.

$$G/N \longrightarrow G/H$$
$$gN \longmapsto gH.$$

This is a surj. homom.

F.I.T.: $(G/N)/\text{kernel} \cong \text{image}$

i.e. $(G/N)/(H/N) \cong G/H.$ ∎

Ex $15\mathbb{Z} \subset 5\mathbb{Z} \subset \mathbb{Z}$

$$\frac{(\mathbb{Z}/15\mathbb{Z})}{(5\mathbb{Z}/15\mathbb{Z})} \cong \mathbb{Z}/5\mathbb{Z}$$