Normal subgroups and their quotients

Thm Let N be a normal subgroup of G. Then the cosets of N in G form a group of order $[G:N]$.

$\vdots$

Def A group is simple if it has no nontrivial, proper, normal subgroup.

Ex. For any prime p, $\mathbb{Z}_p$ is simple.

Thm. For $n \geq 5$, the alternating group A_n is simple.

Exercise. Verify that A_4 is not simple. What about the others?

Properties of homomorphisms

Recall: A homomorphism of groups is a map $\phi: G \to H$ which makes the following diagram commute:

$$\begin{tikzcd} G \times G \arrow[r, "\circ_G"] \arrow[d, "\phi \times \phi"'] & G \arrow[d, "\phi"] \\ H \times H \arrow[r, "\circ_H"] & H \end{tikzcd}$$

The homomorphic image of G under ϕ is $\phi(G) \subseteq H$.

Ex. ① $\phi: \mathbb{Z} \to \mathbb{Z}_n$ is a homom.
$k \mapsto k \pmod n$

② $\phi: GL_n(\mathbb{R}) \to \mathbb{R}^*$ is a homom., since
$A \mapsto \det A$ $\det(AB) = \det(A)\det(B)$.

③ Consider $S^1 \subset \mathbb{C}$ is a group w/ complex mult.
$\phi: \mathbb{R} \to S^1$ is a homom.
$t \mapsto \exp(2\pi i t)$

Prop. Let $\phi: G \to H$ is a homomorphism of groups, then

① $\phi(e_G) = e_H$;

② $\phi(g^{-1}) = (\phi(g))^{-1}$, $\forall g \in G$;

③ if $K \leq G$ is a subgroup of G, then $\phi(K) \subseteq H$ is a subgroup of H;

④ if $K \leq H$ is a subgroup of H, then $\phi^{-1}(K) \leq G$ is a subgroup of G;

⑤ if $N \trianglelefteq H$ is a normal subgroup of H, then $\phi^{-1}(N) \leq G$ is a normal subgroup of G.

(Proof.) ①-③ : exercise

④ Take $K \leq H$ a subgroup. We'll show that $\phi^{-1}(K) \leq G$ is a subgroup.

nonempty: $\phi(e_G) = e_H \in K \Rightarrow e_G \in \phi^{-1}(K)$ ✓

$g_1 g_2^{-1}$: take $g_1, g_2 \in \phi^{-1}(K)$. Then

$$\phi(g_1 g_2^{-1}) = \phi(g_1)\,\phi(g_2^{-1})$$
$$= \phi(g_1)\,(\phi(g_2))^{-1} \in K,$$

since $\phi(g_1), \phi(g_2) \in K$ & K is a subgroup.

$\therefore\ g_1 g_2^{-1} \in \phi^{-1}(K)$. ✓

5 If $N \leq H$ is a normal subgroup, then 4 says that $\phi^{-1}(N) \leq G$ is a subgroup. Take any $g \in G$. We WTS $g\,\phi^{-1}(N)\,g^{-1} = \phi^{-1}(N)$.

Pick $g' \in \phi^{-1}(N)$. Then $\phi(g') \subseteq N$, so

$$\phi(g g' g^{-1}) = \phi(g)\,\phi(g')\,\phi(g^{-1})$$
$$= \phi(g)\,\phi(g')\,(\phi(g))^{-1} \in N,$$

since N is normal. So $g g' g^{-1} \in \phi^{-1}(N)$.

$$\therefore\ g\,\phi^{-1}(N)\,g^{-1} \subseteq \phi^{-1}(N). \qquad \blacksquare$$