Cayley's theorem

Every group is isomorphic to a group of permutations.

(Proof sketch.) Define a map

$$\lambda : G \longrightarrow S_G$$

$$g \mapsto \underbrace{(h \mapsto gh)}_{\lambda_g : G \to G.}$$

left regular representation

Check: λ is an isomorphism of G onto its image. ▨

Direct products

Def. The external direct product $(G \times H, \circ_{G\times H})$ of the groups $(G, \circ_G)$ & $(H, \circ_H)$ consists of the set

$$G \times H = \{(g,h) \mid g \in G,\ h \in H\}$$

and the operation $\circ_{G\times H}$ defined by

$$(g_1, h_1) \circ_{G\times H} (g_2, h_2) := (g_1 \circ_G g_2,\ h_1 \circ_H h_2).$$

Fact. The external direct product is a group.

Similarly define $\prod_{i=1}^{n} G_i$ from $G_1, G_2, \ldots, G_n$.

Thm Let $G = \prod_{i=1}^{n} G_i$, and suppose $g_i \in G_i$ has order $r_i \geq 1$, for $1 \leq i \leq n$. Then $(g_1, g_2, \ldots, g_n) \in G$ has order

$$\operatorname{lcm}(r_1, \ldots, r_n).$$

Thm Let $n_1, n_2, \dots, n_k$ be positive integers. Then $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ is isomorphic to $\mathbb{Z}_{n_1 n_2 \cdots n_k}$ iff $\gcd(n_i, n_j) = 1$ for each $1 \leq i \neq j \leq k$.

Cor For any positive integer m, $\mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$, where $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is the prime factorization of m.

Def. Let $H, K \leq G$ be subgroups such that

1. $H \cap K = \{e\}$;
2. $hk = kh$, $\forall h \in H$ & $k \in K$.

Then the internal direct product of H and K is

$$HK := \{hk \mid h \in H, k \in K\}.$$

Prop. When defined, $HK \leq G$ is a subgroup.

Ex. Consider $\overline{G} = \{(g, e_H) \mid g \in G\} \leq G \times H$
& $\overline{H} = \{(e_G, h) \mid h \in H\} \leq G \times H$.

Check: $\overline{G}\,\overline{H}$ is defined, and in fact $\overline{G}\,\overline{H} = G \times H$.

Thm. Let $H, K \leq G$ be subgroups for which the internal direct product is defined. Then $HK \cong H \times K$.

Normal subgroups

Given a subgroup $H \leq G$, the collection of left (right) cosets of H in G tells us what's left over when we treat elements of H as if they were the identity.

Ex. $H = 3\mathbb{Z} < \mathbb{Z} = G$. If we think of G as acting on a triangle by rotation, then the elements of H do nothing. The (left) cosets of $H \leq G$ are

$$0 + 3\mathbb{Z}, \quad 1 + 3\mathbb{Z}, \quad 2 + 3\mathbb{Z}.$$

These represent the three symmetries actually induced by G on the triangle.

We really want the collection of left (right) cosets of $H \leq G$ to be a group. In general, this will fail.

Scratch / nonsense:

We want a group operation like

$$(g_1 H)(g_2 H) := (g_1 g_2) H$$

on the collection of cosets. In this setup, H should be the identity element, so we want something like $\underbrace{gH = Hg}_{\text{This is not always true!}}$.

Def. A subgroup $N \subseteq G$ is called **normal** if
$gN = Ng$, for every $g \in G$.
i.e., the left cosets = the right cosets

Remark. This definition is tailor-made to ensure that the collection of cosets of N in G has a group structure.

Ex

1. Every subgroup of an abelian group is normal.
2. We previously found that the left & right cosets of $H = \{(1), (12)\} \subset S_3$ are not equal. So H is not a normal subgroup.
3. $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$ is normal, by determinant properties. Namely, $\forall\ A \in GL_n(\mathbb{R})$

$$A \cdot SL_n(\mathbb{R}) = \{M \in GL_n(\mathbb{R}) \mid \det M = \det A\}$$
$$SL_n(\mathbb{R}) \cdot A = \{M \in GL_n(\mathbb{R}) \mid \det M = \det A\}.$$

Thm Let $N \subseteq G$ be a subgroup of G. Then TFAE:

1. N is a normal subgroup;
2. $gNg^{-1} \subseteq N$, $\forall g \in G$;
3. $gNg^{-1} = N$, $\forall g \in G$.