Math 4803 Midterm

March 4, 2024

Name: ____________________ **gtID**: ____________________

Instructions. Read each question carefully and show all your work. Answers without justification will receive little to no credit. Writing your answers in a legible, well-organized manner will maximize your opportunities for partial credit.

Part of the grade for proof-based problems will depend on the quality of your writing. **Proofs should be written in complete sentences.**

This is a closed-note, closed-book exam, and you are expected to abide by the Georgia Tech Honor Challenge. Good luck!

Clearly label any extra papers you want graded.

By signing below, I certify that all work submitted on this exam is my own, and that I have neither given nor received any unauthorized help on this exam.

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1. Complete the following definitions:

   (a) (4 points)
   A **semi-metric** on a set X is a function $d\colon X \times X \to \mathbb{R}$ such that for all $P, Q, R \in X$,

   (1) $d(P,Q) \geq 0$ and $d(P,P) = 0$;

   (2) $d(Q,P) = d(P,Q)$;

   (3) $d(P,R) \leq d(P,Q) + d(Q,R)$.

   A semi-metric which additionally satisfies $d(P,Q) = 0$ if and only if $P = Q$ for all $P, Q \in X$ is called a **metric**.

   (b) (2 points)
   Let d be a path metric on X. A **geodesic** in the metric space (X,d) is a curve γ in X such that for every $P \in \gamma$ there exists $\epsilon > 0$ with the property that if $Q_0, Q_1 \in \gamma$ satisfy $d(P,Q_0) < \epsilon$ **and** $d(P,Q_1) < \epsilon$, then **the section of** $\gamma \cap B_d(P,\epsilon)$ **joining** Q_0 **to** Q_1 **has length** $d(Q_0,Q_1)$.

   (c) (2 points)
   A **linear fractional map** is a **nonconstant** map φ of the form $\varphi(z) = \dfrac{az+b}{cz+d}$ with complex coefficients $a, b, c, d \in \mathbb{C}$.

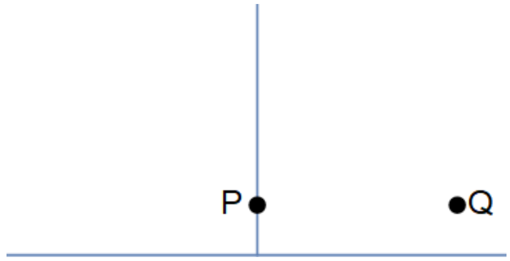
   (d) (2 points)
   Let (X,d) be a metric space, and let $\overline{X}$ be a partition of X. If $\overline{P}$ and $\overline{Q}$ are two points of $\overline{X}$ corresponding to P and Q in X, respectively, then a **discrete walk** w from $\overline{P}$ to $\overline{Q}$ is a finite sequence

$$P = P_1, Q_1, P_2, Q_2, P_3, \ldots, Q_{n-1}, P_n, Q_n = Q$$

   of points of X such that $\overline{Q}_i = \overline{P}_{i+1}$, for every $1 \leq i < n$. The **length** of w is defined to be $\ell_d(w) := \sum_{i=1}^{n} d(P_i, Q_i)$.

2. In the hyperbolic plane $\mathbb{H}^2$, consider the two points $P = i$ and $Q = 4 + i$. For $\alpha \geq 1$, let $P_\alpha = \alpha\, i$, let $Q_\alpha = 4 + \alpha\, i$, and let γ_α be the curve going from P to Q that is made up of the vertical line segment $[P, P_\alpha]$, followed by the horizontal line segment $[P_\alpha, Q_\alpha]$, and finally followed by the vertical line segment $[Q_\alpha, Q]$.

   (a) (1 point) Draw a picture of γ_α, for some $\alpha > 1$.



> **Solution:** Here's the $\alpha = 3$ picture:
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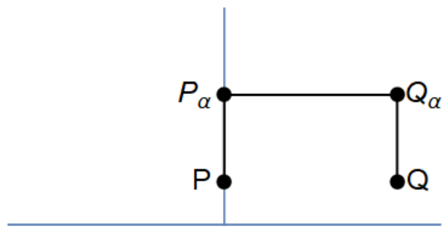
> 
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>
> Any other $\alpha > 1$ will lead to a similar plot.

   (b) (4 points) Compute the hyperbolic length $\ell_{\text{hyp}}(\gamma_\alpha)$. You may use the following parametrizations without justification:

$$\begin{aligned} [P, P_\alpha] &: t \mapsto (0, t), \quad 1 \leq t \leq \alpha \\ [P_\alpha, Q_\alpha] &: t \mapsto (t, \alpha), \quad 0 \leq t \leq 4 \\ [Q_\alpha, Q] &: t \mapsto (4, 1 + \alpha - t), \quad 1 \leq t \leq \alpha. \end{aligned}$$

> **Solution:** Recall that a curve in $\mathbb{H}^2$ with parametrization $t \mapsto (x(t), y(t))$, $a \leq t \leq b$ has hyperbolic length $\int_a^b (\sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2})/y\, dt$. The lengths of the three segments are given by:
>
> $$\begin{aligned} \ell_{\text{hyp}}([P, P_\alpha]) &= \int_1^\alpha \sqrt{0^2 + (-1)^2}/(1 + \alpha - t)\, dt = -\ln|1 + \alpha - t||_1^\alpha = \ln|\alpha| - \ln|1| = \ln\alpha \\ \ell_{\text{hyp}}([P_\alpha, Q_\alpha]) &= \int_0^4 \sqrt{1^2 + 0^2}/\alpha\, dt = 4/\alpha \\ \ell_{\text{hyp}}([Q_\alpha, Q]) &= \int_1^\alpha \sqrt{0^2 + 1^2}/t\, dt = \ln|t||_1^\alpha = \ln|\alpha| - \ln|1| = \ln\alpha. \end{aligned}$$
>
> Altogether, we find that $\boxed{\ell_{\text{hyp}}(\gamma_t) = 2\,\ln\alpha + 4/\alpha}$.

This problem continues on the next page!

(c) (2 points) For what value of α is $\ell_{\mathrm{hyp}}(\gamma_\alpha)$ minimum? Fully justify your claim.
Hint: Treat $\ell_{\mathrm{hyp}}(\gamma_\alpha)$ as a function of α and use your optimization skills from calculus.

Solution: We can optimize the function $f(\alpha) := \ell_{\mathrm{hyp}}(\gamma_\alpha)$ by first taking its derivative:

$$f'(\alpha) = \frac{d}{d\alpha}(2\ \ln\alpha + 4/\alpha) = \frac{2}{\alpha} - \frac{4}{\alpha^2} = \frac{2\alpha - 4}{\alpha^2}.$$

Notice that $f'(\alpha) = 0$ if and only if $\alpha = 2$, and that

$$f''(\alpha) = \frac{(2)(\alpha^2) - (2\alpha - 4)(2\alpha)}{\alpha^4} = \frac{8\alpha - 2\alpha^2}{\alpha^4} = \frac{8 - 2\alpha}{\alpha^3}.$$

Since $f''(2) > 0$, we conclude that f has a global minimum at $\boxed{\alpha = 2}$, and thus the minimum value of $\ell_{\mathrm{hyp}}(\gamma_\alpha)$ is $f(2) = 2\ \ln 2 + 2$.

(d) (3 points) Use the previous part to show that $d_{\mathrm{hyp}}(P, Q) \leq 2\ \ln 2 + 2$.

Solution: Recall that d_{hyp} is a path metric, defined by

$$d_{\mathrm{hyp}}(P, Q) := \inf\{\ell_{\mathrm{hyp}}(\gamma_\alpha) \mid \gamma \text{ is a p.w.d. path in } (\mathbb{H}^2, d_{\mathrm{hyp}}) \text{ connecting } P \text{ to } Q\}.$$

Since γ_2 is a piecewise differentiable curve in $(\mathbb{H}^2, d_{\mathrm{hyp}})$ connecting P to Q, we find that

$$d_{\mathrm{hyp}}(P, Q) \leq \ell_{\mathrm{hyp}}(\gamma_2) = 2\ \ln 2 + 2,$$

as desired.

3. Throughout this problem, consider three pairwise distinct points $z_0, z_1, z_\infty \in \mathbb{C}$.

   (a) (4 points) Recall that $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Construct a linear fractional map $\varphi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying

$$\varphi(z_0) = 0, \quad \varphi(z_1) = 1, \quad \text{and} \quad \varphi(z_\infty) = \infty.$$

   Your final answer should be expressed in the form $\varphi(z) = \dfrac{az+b}{cz+d}$, for some $a, b, c, d \in \mathbb{C}$.

Solution: We can declare that

$$\varphi(z) := \frac{z - z_0}{z - z_\infty} \cdot \frac{z_1 - z_\infty}{z_1 - z_0}$$

and see that φ has the desired properties. We then put this into the desired form:

$$\boxed{\varphi(z) = \frac{(z_1 - z_\infty)z - z_0(z_1 - z_\infty)}{(z_1 - z_0)z - z_\infty(z_1 - z_0}}.$$

   (b) (3 points) Verify that the linear fractional map φ constructed in part (a) satisfies $ad - bc \neq 0$.

Solution: We have $a = z_1 - z_\infty$, $b = z_0(z_\infty - z_1)$, $c = z_1 - z_0$, and $d = z_\infty(z_0 - z_1)$, so

$$ad - bc = (z_1 - z_\infty)z_\infty(z_0 - z_1) - z_0(z_\infty - z_1)(z_1 - z_0) = (z_0 - z_1)(z_1 - z_\infty)(z_\infty - z_0).$$

Since each of the differences $z_0 - z_1$, $z_1 - z_\infty$, and $z_\infty - z_0$ is nonzero (the points z_0, z_1, and z_∞ being distinct), their product is nonzero.

   (c) (3 points) Show that if $\tilde{\varphi}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is any linear fractional map satisfying

$$\tilde{\varphi}(z_0) = 0, \quad \tilde{\varphi}(z_1) = 1, \quad \text{and} \quad \tilde{\varphi}(z_\infty) = \infty,$$

   then $\tilde{\varphi}(z) = \varphi(z)$, for all $z \in \widehat{\mathbb{C}}$.
   Hint: You (basically) showed in a homework exercise that if ψ is a linear fractional map which is not the identity map, then ψ has at most two fixed points. Use this.

Solution: Our computation in part (b) tells us that φ is invertible, and thus we may consider $\varphi^{-1}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Notice, then, that

$$(\varphi^{-1} \circ \tilde{\varphi})(z_0) = \varphi^{-1}(\tilde{\varphi}(z_0)) = \varphi^{-1}(0) = z_0.$$

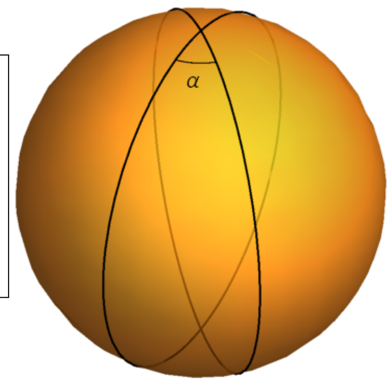
We similarly find that $(\varphi^{-1} \circ \tilde{\varphi})(z_1) = z_1$ and $(\varphi^{-1} \circ \tilde{\varphi})(z_\infty) = z_\infty$. So $\varphi^{-1} \circ \tilde{\varphi}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a linear fractional map with at least three fixed points, and therefore must be the identity map. That is, $(\varphi^{-1} \circ \tilde{\varphi})(z) = z$, and thus $\tilde{\varphi}(z) = \varphi(z)$, for all $z \in \widehat{\mathbb{C}}$.

4. Throughout this problem, let S^2 denote the unit sphere with its usual metric d_{sph}.

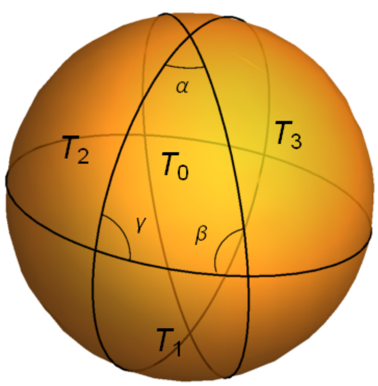
   (a) (3 points) Consider great circles B and C on S^2 which intersect at an angle of $0 < \alpha \leq \pi/2$. Show that the surface area of the (smaller) digon bounded by B and C is equal to 2α.
   Hint: You may use without proof that S^2 has surface area 4π.

Solution: We can argue by proportionality. Because the total angle around an intersection point is 2π, the ratio of the area of the digon to the area of the sphere will be $\alpha/2\pi$. Thus the area of the digon is

$$\frac{\alpha}{2\pi} \cdot 4\pi = \boxed{2\alpha}.$$



   (b) (3 points) Let T_0 be a spherical triangle with angles α, β, and γ, and let A, B, and C be the great circles of S^2 containing the three edges of S^2, as pictured. Let T_1 be the triangle sharing an edge along A with T_0, T_2 the triangle sharing an edge along B with T_0, and T_3 the triangle sharing an edge along C with T_0. Produce a system of four linear equations in the four variables $A_i := \text{Area}(T_i)$. **Justify your equations with words.**
   Hint: For one of your equations, it will be helpful to use the fact that the antipodal map is a spherical isometry.



Solution: Notice that T_0 and T_1 partition the digon bounded by B and C, and thus $A_0 + A_1 = 2\alpha$. Similarly, the digons bounded by A and C and by A and B are partitioned as $T_0 \cup T_2$ and $T_0 \cup T_3$, respectively, giving us the equations $A_0 + A_2 = 2\beta$ and $A_0 + A_3 = 2\gamma$. Finally, A, B, and C partition S^2 into 8 triangles: $T_0, T_1, T_2, T_3, \phi(T_0), \phi(T_1), \phi(T_2)$, and $\phi(T_3)$, where φ is the antipodal map. Thus we find that

$$4\pi = 2A_0 + 2A_1 + 2A_2 + 2A_3,$$

since $\text{Area}(\varphi(T_i)) = \text{Area}(T_i) = A_i$. Altogether, we have the system

$$\boxed{\begin{aligned} A_0 + A_1 &= 2\alpha \\ A_0 + A_2 &= 2\beta \\ A_0 + A_3 &= 2\gamma \\ 2A_0 + 2A_1 + 2A_2 + 2A_3 &= 4\pi \end{aligned}}.$$

This problem continues on the next page!

(c) (2 points) Solve the previous system for A_0 in terms of α, β, and γ.

Solution: Summing the first three equations yields $3A_0 + A_1 + A_2 + A_3 = 2(\alpha + \beta + \gamma)$, and thus
$$A_1 + A_2 + A_3 = 2(\alpha + \beta + \gamma) - 3A_0.$$
Substituting this into the last equation gives us
$$4\pi = 2A_0 + 2(2(\alpha + \beta + \gamma) - 3A_0) = 4(\alpha + \beta + \gamma) - 4A_0.$$
Solving this equation for A_0 gives $\boxed{A_0 = \alpha + \beta + \gamma - \pi}$.

(d) (2 points) Show that we must have $\pi < \alpha + \beta + \gamma < \pi + 2\,\min\{\alpha, \beta, \gamma\}$.
Hint: First show that $\pi < \alpha + \beta + \gamma < \pi + 2\alpha$. An analogous argument will work for β and γ.

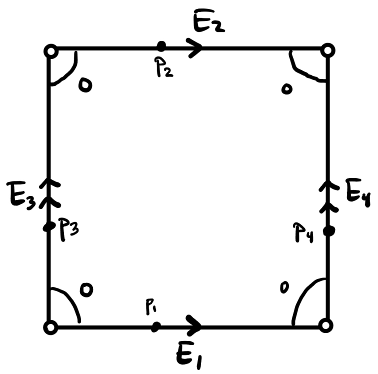
Solution: Certainly we have $0 < A_0 = \alpha + \beta + \gamma - \pi$. Additionally, since $A_0 + A_1 = 2\alpha$ we see that $A_0 = 2\alpha - A_1$. But the triangle T_1 has positive area, so in fact $A_0 < 2\alpha$. We similarly find that $A_0 < 2\beta$ and $A_0 < 2\gamma$. That is,
$$0 < A_0 < \min\{2\alpha, 2\beta, 2\gamma\}.$$
We may add π to each portion of the inequality to yield
$$\pi < \alpha + \beta + \gamma < \pi + 2\,\min\{\alpha, \beta, \gamma\},$$
as desired.

5. Each figure in this problem is a schematic for a polygon in the Euclidean plane $(\mathbb{R}^2, d_{\mathrm{euc}})$, the hyperbolic plane $(\mathbb{H}^2, d_{\mathrm{hyp}})$, or the sphere (S^2, d_{sph}), along with an edge gluing of the polygon. The figures are not intended to be geometrically accurate, but some contain geometric information, and each figure is compatible with at least one of the three fundamental geometries.

   For each figure, determine the homeomorphism type of the Euclidean, hyperbolic, or spherical surface that results from the given edge gluing. Additionally, determine which of the three fundamental geometries is compatible with the figure. **One of the figures is compatible with more than one geometry!**

   This question is not intended to trick you. Think only about homeomorphism types and geometric structures that we've seen in class, and don't worry about proving the (in)compatibility of a figure with a geometry.

   Here's an example:

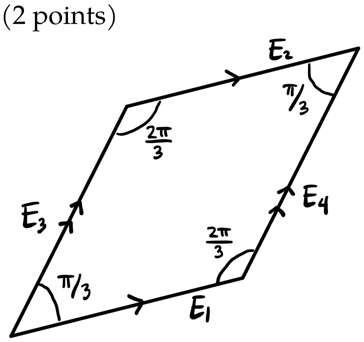


homeomorphism type: **punctured torus**

□ Euclidean
√ **hyperbolic**
□ spherical

Note: The punctured torus admits a Euclidean metric, but we've only seen the angle measurements in the figure when we put a hyperbolic metric on the punctured torus.

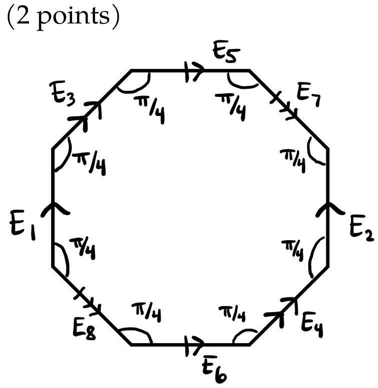
(a) (2 points)



homeomorphism type: **torus**

√ **Euclidean**
□ hyperbolic
□ spherical

(b) (2 points)



homeomorphism type: **surface of genus two**

□ Euclidean
√ **hyperbolic**
□ spherical

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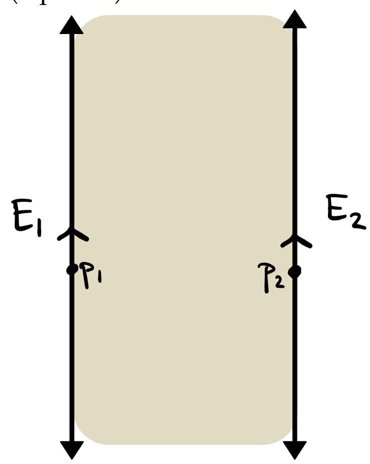
(c) (2 points)



homeomorphism type: **cylinder**

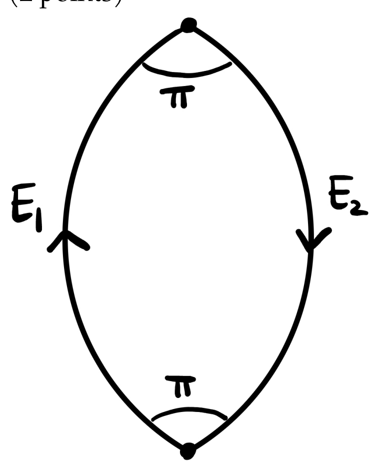
√ **Euclidean**

√ **hyperbolic**

□ spherical

Note: The solid arrows indicate that this polygon is unbounded, while the edge gluing is determined by the decorations. In particular, $\varphi_1(p_1) = p_2$.

(d) (2 points)



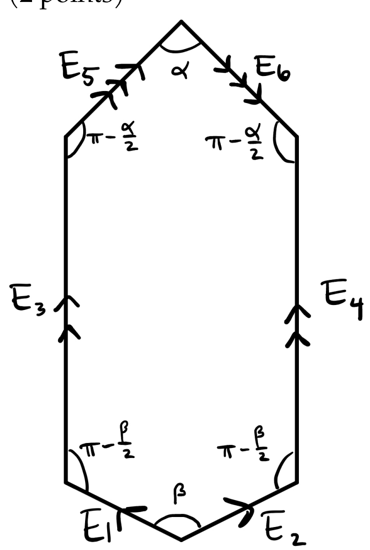
homeomorphism type: **projective plane**

□ Euclidean

□ hyperbolic

√ **spherical**

(e) (2 points)



homeomorphism type: **Klein bottle**

√ **Euclidean**

□ hyperbolic

□ spherical

Hint: Consider cutting this polygon with a vertical line.