Math 4803 March 6, 2024

LAST TIME

We defined tessellations and stated the tessellation theorem.

TODAY

We'll outline the proof of the tessellation theorem.

Tessellations

Let X be the Euclidean plane, the hyperbolic plane, or the sphere.

A tessellation of X is a family of tiles X_n, $n \in \mathbb{N}$ such that

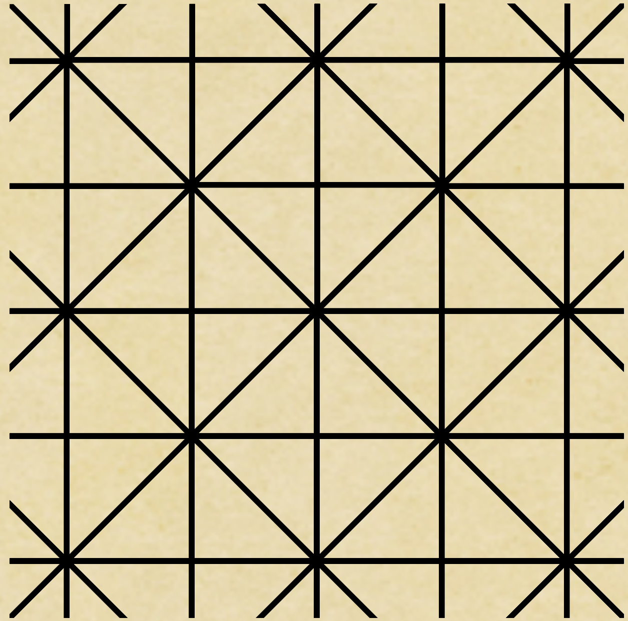
(1) each tile is a connected polygon in X;

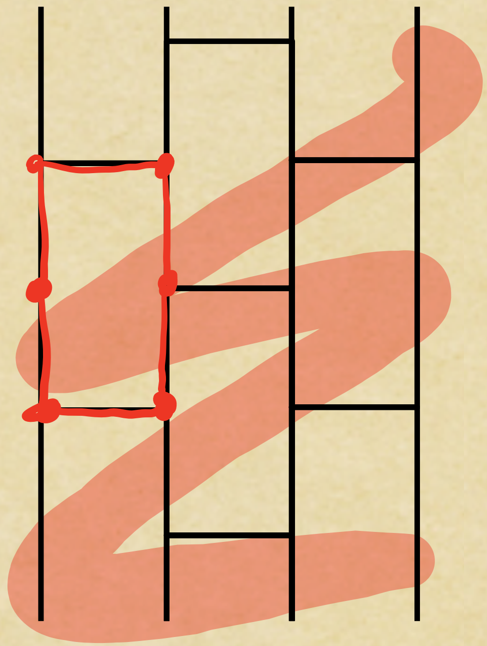
(2) the tiles are pairwise isometric;

(3) the union of the tiles is X;

(4) for $m \neq n$, $X_m \cap X_n$ consists only of vertices & edges of X_m, and these are shared with X_n.

(5) for every $P \in X$, there exists $\varepsilon > 0$ s.t. $\{n \in \mathbb{N} \mid B_d(P, \varepsilon) \cap X_n \neq \emptyset\}$ is finite.





The tessellation theorem

Thm. Let X be a connected polygon in the Euclidean plane, hyperbolic plane, or sphere, and suppose that an edge gluing $\{\varphi_i : E_i \to E_{i\pm 1}\}$ has been specified. If

(1) for every vertex $P \in X$, $\sum_{Q \sim P} \measuredangle(Q) = \frac{2\pi}{n}$, where $n > 0$ is an integer which may depend on P;

(2) the quotient metric space $(\overline{X}, \overline{d}_X)$ is complete;

then the family $\{\varphi(X) \mid \varphi \in \Gamma\}$ is a tessellation of the Euclidean plane, hyperbolic plane, or sphere.

Proof strategy

We'll focus on the case where X is a polygon in $(\mathbb{R}^2, d_{euc})$. The hyperbolic version is exactly the same, and the spherical case requires only a small modification.

Verifying that $\{\varphi(X) \mid \varphi \in \Gamma\}$ requires that we check three things:

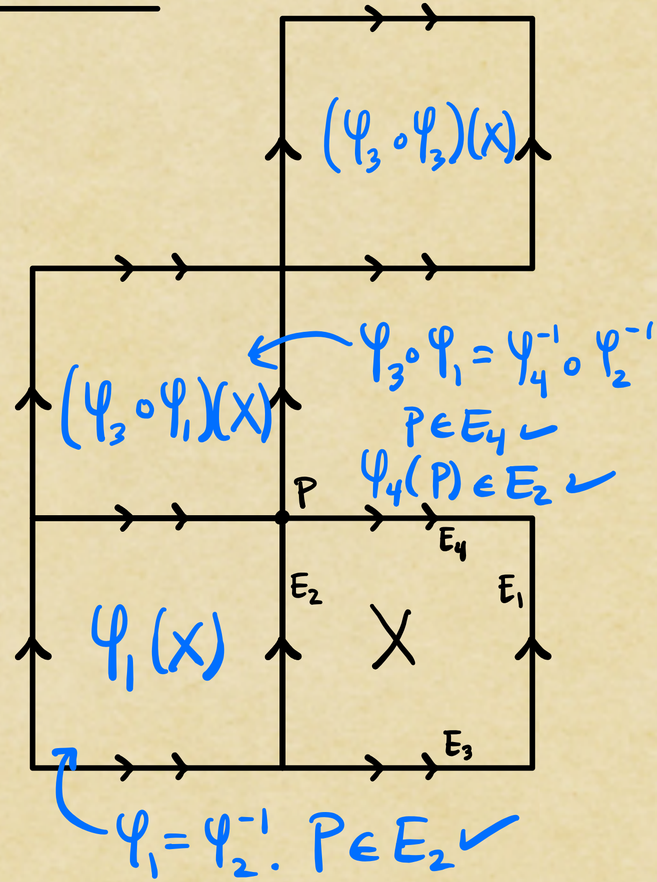
- $\bigcup_{\varphi \in \Gamma} \varphi(X) = \mathbb{R}^2$;
- if $\varphi(X) \neq \tilde{\varphi}(X)$, then $\varphi(X) \cap \tilde{\varphi}(X)$ consists of edges & vertices ;
- local finiteness.

The other properties of tessellations will be easier.

Proof strategy

We'll say that a tile $\varphi(X)$ is adjacent to X at $P \in X \cap \varphi(X)$ if we can write $\varphi = \varphi_{i_1}^{-1} \circ \varphi_{i_2}^{-1} \circ \cdots \circ \varphi_{i_\ell}^{-1}$ for some sequence $i_1, i_2, \ldots, i_\ell$ such that

- $P \in E_{i_1}$;
- for $2 \le j \le \ell$, $(\varphi_{i_{j-1}} \circ \varphi_{i_{j-2}} \circ \cdots \circ \varphi_{i_1})(P)$ is contained in E_{i_j}.



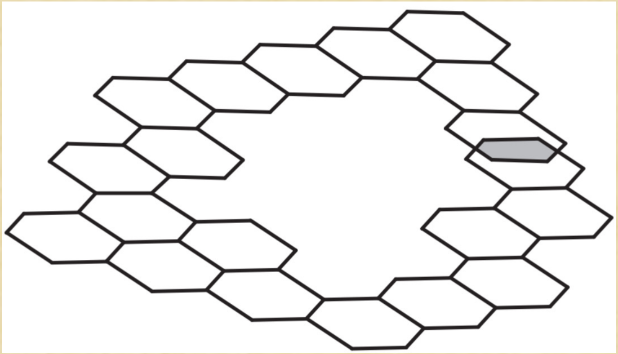
For any $\varphi, \psi \in \Gamma$, we'll call $\varphi(X)$ & $\psi(X)$ adjacent at $P \in \varphi(X) \cap \psi(X)$ if $(\psi^{-1} \circ \varphi)(X)$ is adjacent to X at $\psi^{-1}(P)$.

Proof strategy

Our first crucial lemma is:

Lemma 1 For any $P \in X$, there are finitely many tiles adjacent to X at P.

We'll try to return to the proof later, but the key hypothesis here is that $\sum_{Q \sim P} \measuredangle(Q) = \frac{2\pi}{n}$ if P is a vertex. (Edges & interior points are easy.)



Informally: For any $P \in X$, $\exists\, \varepsilon > 0$ such that the ball $B_d(P, \varepsilon)$ is divided among the tiles adjacent to X @ P.

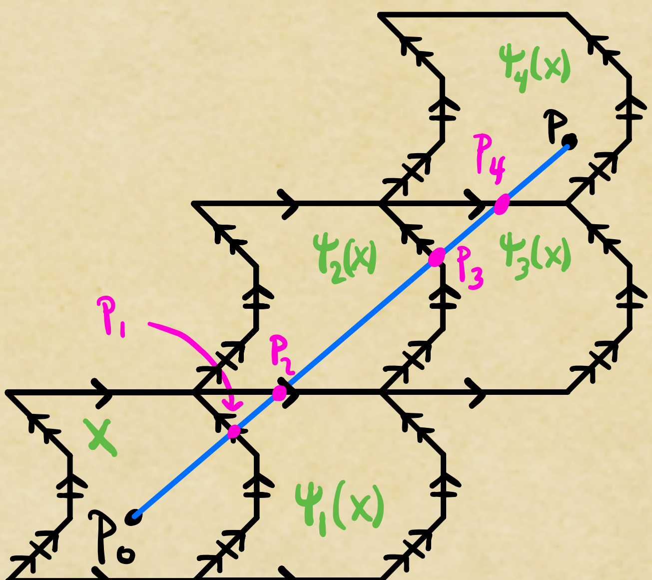
We want this away from X, too.

Proof

Our next lemma directly checks a tessellation property:

Lemma ② $\displaystyle\bigcup_{\varphi\in\Gamma}\varphi(X)=\mathbb{R}^2$

(Proof.) The crucial hypothesis this time is the completeness of $(\bar{X},\bar{d}_X)$. Throughout, let's fix a base point $P_0\in X$. For any $P\in\mathbb{R}^2$, we can let g be the geodesic connecting P_0 to P.



Let $P_1\in\mathbb{R}^2$ be the first point where g leaves X, and let $\psi_1(X)$ be ~~the~~ a tile which g enters. Repeat this to produce points $P_1,P_2,\dots,P_n\in g$ contained in tiles $\psi_1(X),\psi_2(X),\dots,\psi_n(X)$.

Proof

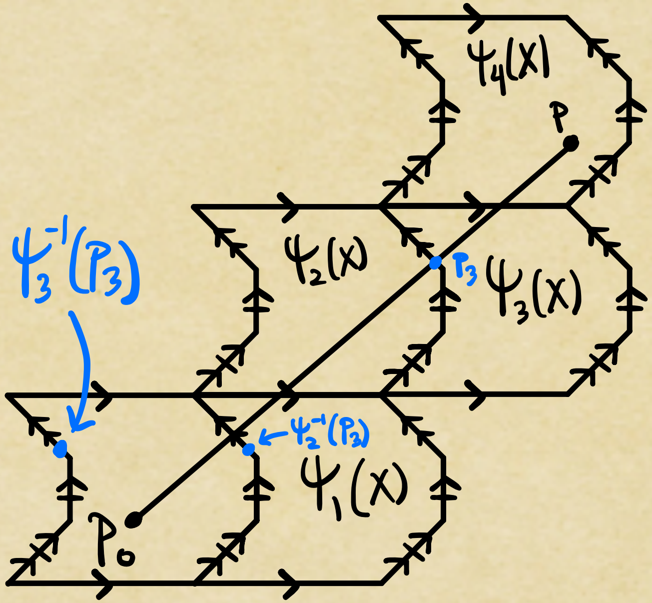
We stop when we reach P and call $\Psi_n(X)$ a <u>canonical tile</u> for $P \in \mathbb{R}^2$.

<u>Claim</u>. This process will terminate, for every $P \in \mathbb{R}^2$.

(Proof of claim.) We'll use the completeness of $(\overline{X}, \overline{d}_X)$.

§ the tiling process does not terminate, giving us $(\Psi_0 = id)$

$$P_k \in \Psi_k(X) \cap \Psi_{k-1}(X), \quad \forall k \geq 1.$$



Then $\Psi_k^{-1}(P_k) \sim \Psi_{k-1}^{-1}(P_k) \in X$. Let's call the common point in $\overline{X}$ $\overline{P_k}$. Then

$$\overline{d}_X(\overline{P_k}, \overline{P_{k+1}}) \leq d_X(\Psi_k^{-1}(P_k), \Psi_k^{-1}(P_{k+1}))$$
$$= d_{\Psi_k(X)}(P_k, P_{k+1})$$
$$= d_{euc}(P_k, P_{k+1}).$$

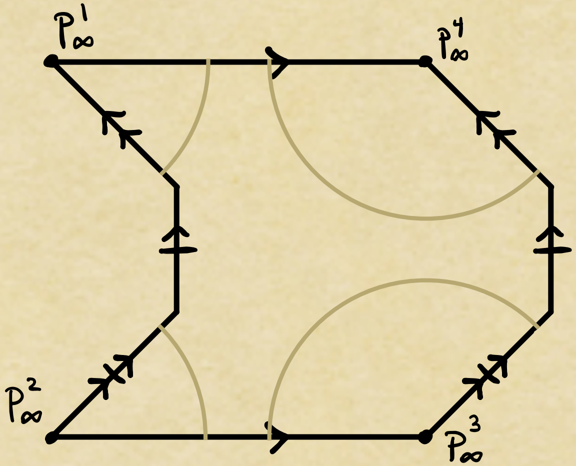
Proof

So $\overline{d}_X(\overline{P}_k, \overline{P}_{k+1}) \leq d_{euc}(P_k, P_{k+1})$, meaning that

$$\sum_{k=1}^{\infty} \overline{d}_X(\overline{P}_k, \overline{P}_{k+1}) \leq \sum_{k=1}^{\infty} d_{euc}(P_k, P_{k+1}) = \ell_{euc}(g) < \infty,$$

This is complete.

and thus the sequence $(\overline{P}_k)$ converges to a point $\overline{P}_\infty \in (\overline{X}, \overline{d}_X)$.



We'll get a bit handwavy now. Let $\overline{P}_\infty = \{P_\infty^1, P_\infty^2, \ldots, P_\infty^m\}$, and fix $\varepsilon > 0$ small enough that $B_{\overline{d}_X}(\overline{P}_\infty, \varepsilon) = \bigsqcup_{i=1}^{m} B_{d_X}(P_\infty^i, \varepsilon)$.

We can pick $k_0 \in \mathbb{N}$ s.t., $\forall\, k \geq k_0$,

$$\overline{P}_k \in B_{\overline{d}_X}(\overline{P}_\infty, \varepsilon/2) \quad \& \quad d_{euc}(P_k, P_{k+1}) < \varepsilon/2 .$$

Together, these ensure that the points $\psi_{k_0}^{-1}(P_{k_0}), \psi_{k_0+1}^{-1}(P_{k_0+1}), \ldots$ are all near the same P_∞^i. In particular, every tile $\psi_k(X)$ is adjacent to $\psi_{k_0}(X)$ at $\psi_{k_0}(P_\infty^i)$, for every $k \geq k_0$.

Upshot: There are finitely many $k \geq k_0$.

◇

Proof strategy

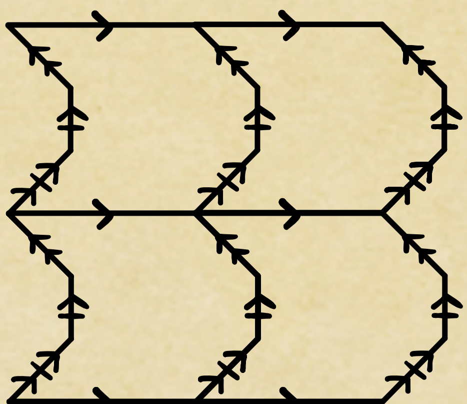
We've now established the claim. The lemma follows b/c each point $P \in \mathbb{R}^2$ is covered by a canonical tile. ◇

Remark: Intuitively, canonical tile for P = tile containing P.

I *think* we're using this language b/c we don't yet know the intersection properties of the tiles.

We now state some lemmas which will let us finish the proof of the tessellation theorem.

Lemma 3 For every $P \in \mathbb{R}^2$, $\exists\, \varepsilon > 0$ s.t., $\forall\, Q \in B_{deuc}(P, \varepsilon)$, the canonical tiles of Q are precisely those canonical tiles of P which contain Q.



Proof strategy

Lemma 4 Let P and Q be in the interior of a tile $\varphi(X)$. If $\varphi(X)$ is canonical for P, then $\varphi(X)$ is canonical for Q and $\varphi(X)$ is the only canonical tile for P and Q.

Lemma 5 Every tile $\varphi(X)$ is canonical for an interior point.

Corollary 6 $P \in \text{interior}(\varphi(X)) \Rightarrow \varphi(X)$ is the unique canonical tile for P.

(Proof of tessellation theorem.) We have five properties to check:

(1) Is every tile connected? Yes, since X is.

(2) Are the tiles pairwise isometric? Yes, by construction.

(3) Do the tiles cover $\mathbb{R}^2$? Yes, by Lemma 2.

(4) Do the tiles intersect correctly? Yes.

Corollary 6 $\Rightarrow$ interiors are disjoint.

Edge gluing $\Rightarrow$ vertex can't be glued to interior of edge.

Proof strategy

(Proof of tessellation theorem.) We have five properties to check:

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Corollary 6 $\Rightarrow$ interiors are disjoint.

Edge gluing $\Rightarrow$ vertex can't be glued to interior of edge.

(5) Are the tiles locally finite? Yes.

Each point $P \in \mathbb{R}^2$ admits a ball $B_{deuc}(P, \varepsilon)$ which meets only its canonical tiles, of which there are finitely many. $\diamond$

Next: Some completeness & compactness properties, and then examples.