Math 4803

March 27, 2024

LAST TIME

Some tessellations of $(\mathbb{H}^2, d_{hyp})$.

- Built via the tessellation theorem.
- Mostly from bounded polygons.

TODAY

We obtain <u>incomplete</u> surfaces as quotients of polygons in $(\mathbb{H}^2, d_{hyp})$.

An incomplete hyperbolic cylinder

Let's revisit and generalize an example where $(\overline{X}, \overline{d})$ is not complete.

Let $X = \{z \in \mathbb{H}^2 \mid 0 \leq \operatorname{Re} z \leq 1\}$, with edges $E_1 = \{\operatorname{Re} z = 0\}$ & $E_2 = \{\operatorname{Re} z = 1\}$.

We'll define a gluing $\varphi_1 : E_1 \to E_2$ s.t. $\varphi_1(0) = 1$

Consider $\varphi_1(z) = \dfrac{az+b}{cz+d}$. $\varphi_1(0) = 1 \Rightarrow b = d$.

$\varphi_1(\infty) = \infty \Rightarrow c = 0$

So we can write $\varphi_1(z) = \left(\frac{a}{d}\right)z + 1$. We need $\operatorname{Im}(\varphi_1(z)) > 0$,
so $0 < \operatorname{Im}(\varphi_1(i)) = \operatorname{Im}\left(\left(\frac{a}{d}\right)i + 1\right) = \dfrac{a}{d}$.

By setting $d = 1$ we can take φ_1 to have the form

$$\varphi_1(z) = az + 1,$$

for some $a > 0$.

An incomplete hyperbolic cylinder

Now the tiling group $\Gamma = \langle \varphi_1, \varphi_2 = \varphi_1^{-1} \rangle$ is cyclic, since we can write $\Gamma = \langle \varphi_1 \rangle$. In particular, any $\varphi \in \Gamma$ can be written $\varphi = \varphi_1^n$, for some $n \in \mathbb{Z}$.

Notice how φ_1 treats vertical lines: if $\operatorname{Re} z = c_0$, then

$$\operatorname{Re}(\varphi_1(z)) = \operatorname{Re}(az+1) = \operatorname{Re}(az) + 1 = a\operatorname{Re}(z) + 1 = \underbrace{ac_0 + 1}_{\text{constant}}$$

So $\varphi_1^n(X)$ is a vertical strip with edges

$$\{z \in \mathbb{H}^2 \mid \operatorname{Re} z = \varphi_1^n(0)\} \;\&\; \{z \in \mathbb{H}^2 \mid \operatorname{Re} z = \underset{\varphi_1^n(1)}{\overset{}{\varphi_1^{n+1}(0)}}\}.$$

Let's find a formula for $\varphi_1^n(0)$.

Prop. $\varphi_1^n(z) = a^n z + \dfrac{1-a^n}{1-a}$, for $n \in \mathbb{Z}$, if $a \neq 1$.

Cor. $\varphi_1^n(0) = \dfrac{1-a^n}{1-a}$, for $n \in \mathbb{Z}$ & $a \neq 1$

An incomplete hyperbolic cylinder

Prop. $\varphi_1^n(z) = a^n z + \dfrac{1-a^n}{1-a}$, for $n \in \mathbb{Z}$, if $a \neq 1$.

(Proof.) We'll proceed by induction on n.

Notice that $\varphi_1^0(z) = z = a^0 z + \dfrac{1-a^0}{1-a}$ ✓.

& the formula holds for $\varphi_1^{n_0}(z)$. We NTS that it holds for n_0+1 & n_0-1.

$$\varphi_1^{n_0+1}(z) = \varphi_1\left(\varphi_1^{n_0}(z)\right) = \varphi_1\left(a^{n_0} z + \frac{1-a^{n_0}}{1-a}\right)$$

$$= a\left(a^{n_0} z + \frac{1-a^{n_0}}{1-a}\right) + 1$$

$$= a^{n_0+1} z + \frac{a - a^{n_0+1}}{1-a} + \frac{1-a}{1-a}$$

$$= a^{n_0+1} z + \frac{1-a^{n_0+1}}{1-a} \quad ✓$$

An incomplete hyperbolic cylinder

Prop. $\varphi_1^n(z) = a^n z + \frac{1-a^n}{1-a}$, for $n \in \mathbb{Z}$, if $a \neq 1$.

(Proof, cont'd)

Now $\varphi_1^{-1}(z) = a^{-1}(z-1)$, so

$$\varphi_1^{n_0-1}(z) = \varphi_1^{-1}\left(\varphi_1^{n_0}(z)\right) = \varphi_1^{-1}\left(a^{n_0}z + \frac{1-a^{n_0}}{1-a}\right)$$

$$= a^{-1}\left(a^{n_0}z + \frac{1-a^{n_0}}{1-a} - 1\right)$$

$$= a^{n_0-1}z + \frac{a^{-1}-a^{n_0-1}}{1-a} - \frac{1-a}{1-a}\cdot a^{-1}$$

$$= a^{n_0-1}z + \frac{1-a^{n_0-1}}{1-a} \checkmark$$

By the principle of mathematical induction, the formula holds for all $n \in \mathbb{Z}$. $\diamond$

An incomplete hyperbolic cylinder

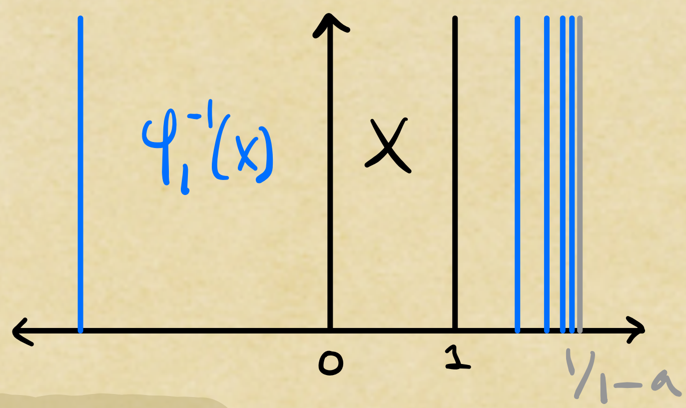
So our tessellation is determined by the sequence

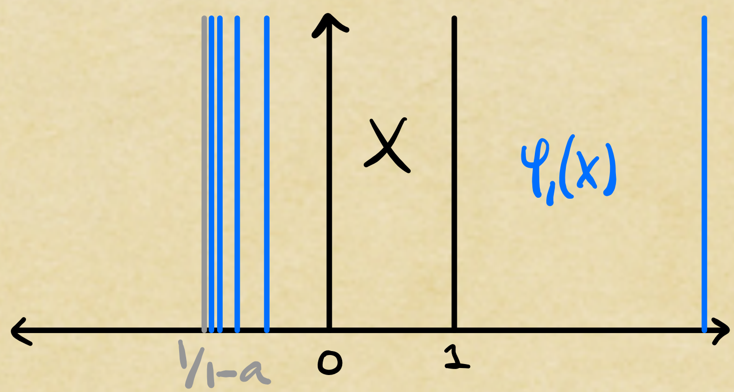
$$a_n := \varphi_1^n(0) = \frac{1-a^n}{1-a}.$$

The behavior of this sequence has two cases:

$a<1$: $\displaystyle\lim_{n\to\infty} a_n = \frac{1}{1-a}$ ⋮ $\displaystyle\lim_{n\to-\infty} a_n = -\infty$

$a>1$: $\displaystyle\lim_{n\to\infty} a_n = \infty$ ⋮ $\displaystyle\lim_{n\to-\infty} a_n = \frac{1}{1-a}$

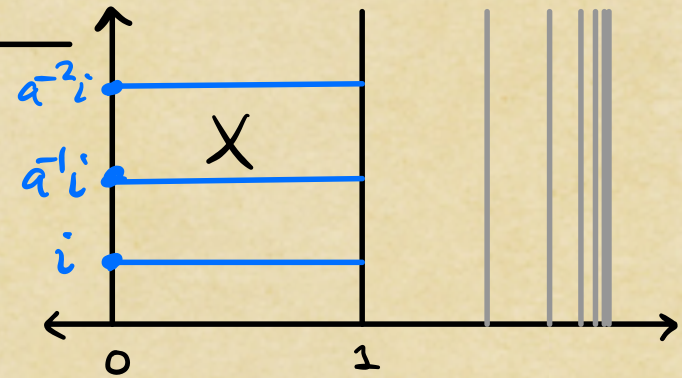




Conclusion: If $a \neq 1$, then the quotient $(\bar{X}, \bar{d})$ is <u>not complete</u>. (If it were, we'd get a tessellation.)

An incomplete hyperbolic cylinder

We can realize the incompleteness of $(\overline{X},\overline{d})$ more directly by constructing a finite-length sequence $(\overline{P}_n)_{n\in\mathbb{N}}$ in $(\overline{X},\overline{d})$ which does not converge in $(\overline{X},\overline{d})$.



WLOG, let's assume that $a<1$.

Consider the sequence $P_n = a^{-n} i$ in X.

Each P_n is glued to $P_n' = a^{-n+1} i + 1$.

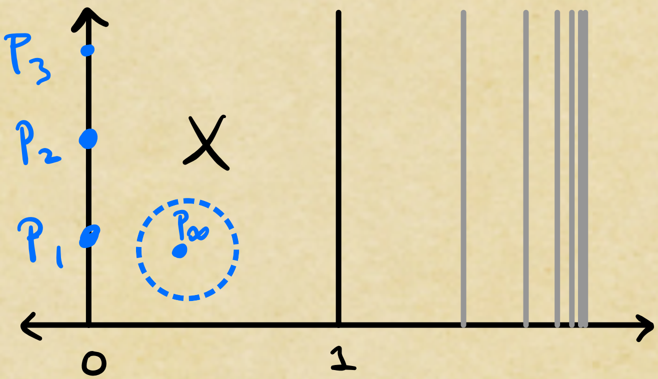
Then
$$\ell((\overline{P}_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \overline{d}(\overline{P}_n, \overline{P}_{n+1}) = \sum_{n=1}^{\infty} \overline{d}(\overline{P}_n, \overline{P}'_{n+1})$$
$$\leq \sum_{n=1}^{\infty} d(P_n, P'_{n+1}) = \sum_{n=1}^{\infty} d(a^{-n}i, a^{-n}i+1)$$

But P_n' & P_{n+1} can be connected by the horizontal line segment $\{z\in\mathbb{H}^2 \mid \operatorname{Im} z = a^{-n},\ 0\leq \operatorname{Re} z \leq 1\}$, which has length a^n. $\leftarrow \frac{1}{y}$

So $\ell((\overline{P}_n)_{n\in\mathbb{N}}) \leq \sum_{n=1}^{\infty} d(a^{-n}i, a^{-n}i+1) \leq \sum_{n=1}^{\infty} a^n = \frac{a}{1-a} < \infty$.

An incomplete hyperbolic cylinder

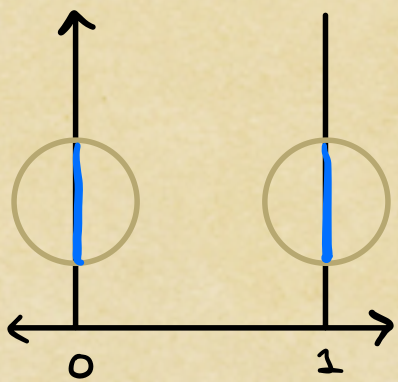
So $(\overline{P_n})_{n\in\mathbb{N}}$ is a finite-length sequence in $(\overline{X},\overline{d})$. We claim that it doesn't converge in $(\overline{X},\overline{d})$.



Suppose $(\overline{P_n})_{n\in\mathbb{N}}$ converges to $\overline{P_\infty}\in(\overline{X},\overline{d})$.

If $P_\infty \in \operatorname{int}(X)$, we can choose $\varepsilon>0$ s.t. $B_d(P_\infty,\varepsilon)$ contains none of the points P_n or P_n'.
But then $B_{\overline{d}}(\overline{P_\infty},\varepsilon)$ contains none of the points $\overline{P_n}$. $\times$

If $P_\infty \in X-\operatorname{int}(X)$, we can choose $\varepsilon>0$ s.t.
$\{P\in E_1\cup E_2 \mid \overline{P}\in B_{\overline{d}}(\overline{P_\infty},\varepsilon)\}$ = two vertical arcs of Euclidean length $2\varepsilon'$



and, in particular, s.t. $B_{\overline{d}}(\overline{P_\infty},\varepsilon)$ contains (n)one of the points $\overline{P_n}$. $\times$

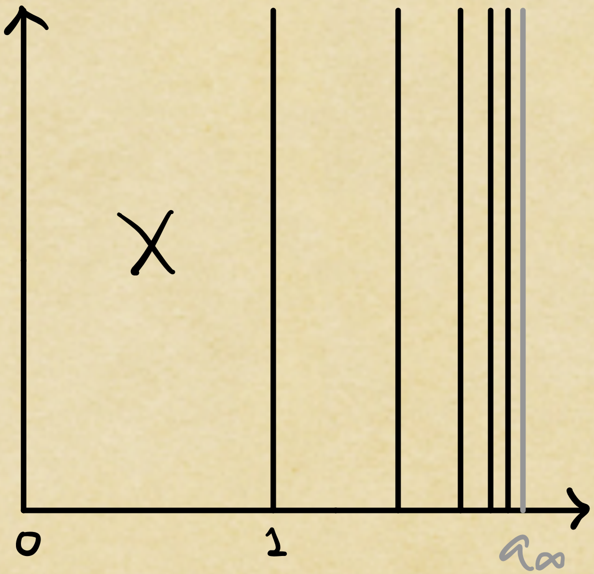
So $(\overline{P_n})_{n\in\mathbb{N}}$ doesn't converge and we again see that $(\overline{X},\overline{d})$ is not complete.

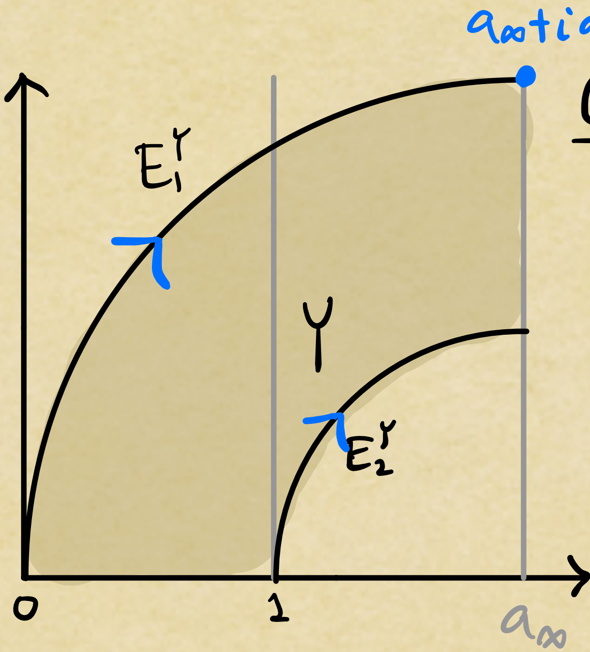
An incomplete hyperbolic cylinder

Finally, let's explore the geometry of $(\bar{X}, \bar{d})$ a bit more.

Fix $0 < a < 1$ and let $a_\infty = \frac{1}{1-a}$. Consider

$$Y = \{ z \in \mathbb{H}^2 \mid \operatorname{Re} z < a_\infty \text{ \& } a_\infty - 1 \le |z - a_\infty| \le a_\infty \}.$$





<u>Claim</u>. $\varphi_1(z) = az + 1$ gives an edge gluing for Y.

(Proof.) Note that

$$\varphi_1(a_\infty) = a_\infty,$$

so $a \cdot a_\infty + 1 = a_\infty$

$a_\infty - 1 = a \cdot a_\infty$.

So if $|z - a_\infty| = a_\infty$, then

$$|\varphi_1(z) - a_\infty| = |\varphi_1(z) - \varphi_1(a_\infty)| = |(az+1) - (a \cdot a_\infty + 1)| = a|z - a_\infty|$$
$$= a \cdot a_\infty = a_\infty - 1$$

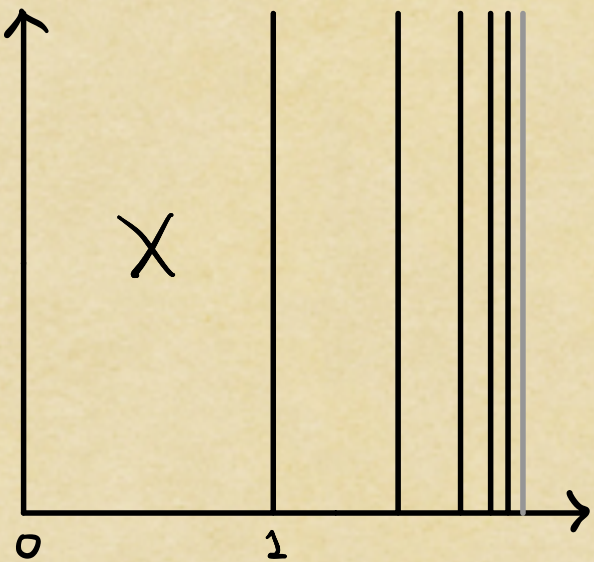
Also, $\varphi_1(a_\infty + i \cdot a_\infty) = a_\infty + 1$. (OR $\varphi_1(0) = 1$)

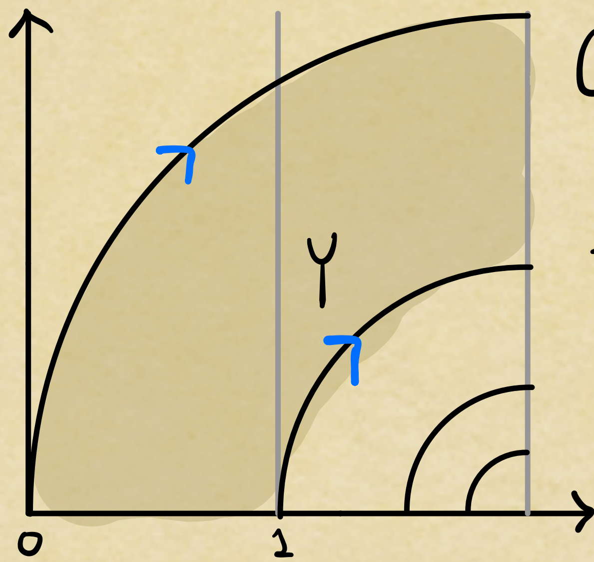
$\diamond$

An incomplete hyperbolic cylinder

Maybe it's believable that both X and Y tessellate

$H = \{ z \in \mathbb{H}^2 \mid \operatorname{Re} z < a_\infty \}$ (with the same tiling group !!)





On H we can define a partition via

$$\overline{P} = \{ \varphi_1^n(P) \mid n \in \mathbb{Z} \}$$

for every $P \in H$ and consider the quotient $(\overline{H}, d_{hyp})$.

Fact: The maps $\overline{X} \to \overline{H}$; $\overline{Y} \to \overline{H}$

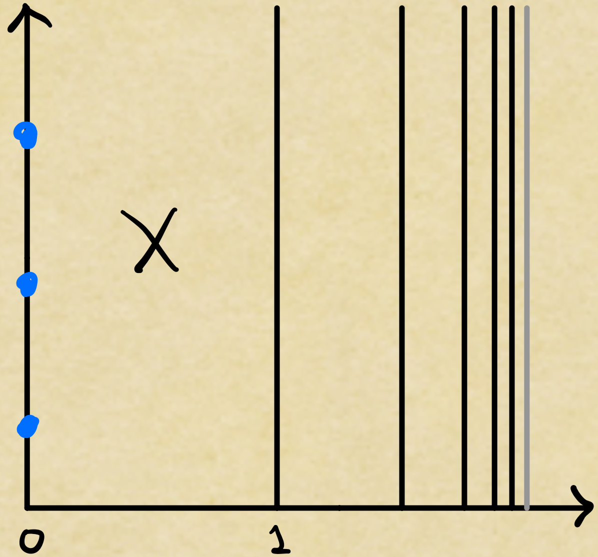
$$\overline{P} \mapsto \overline{P} \qquad \overline{P} \mapsto \overline{P}$$

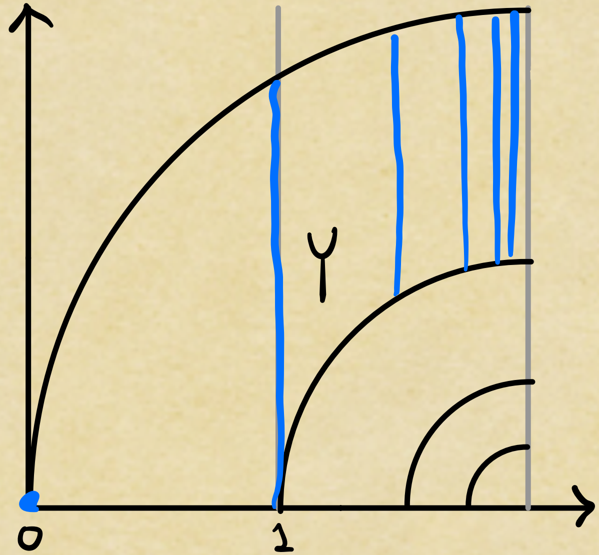
are isometries.

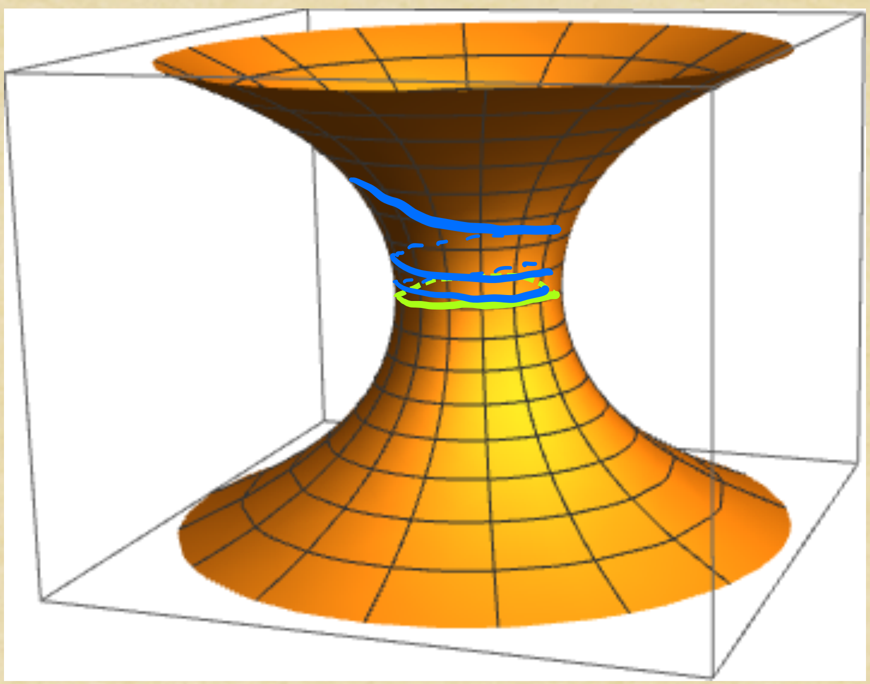
Consequence: $\overline{X}$ and $\overline{Y}$ are isometric to each other

But we recognize $\overline{Y}$ as a hyperbolic half-cylinder !

An incomplete hyperbolic cylinder





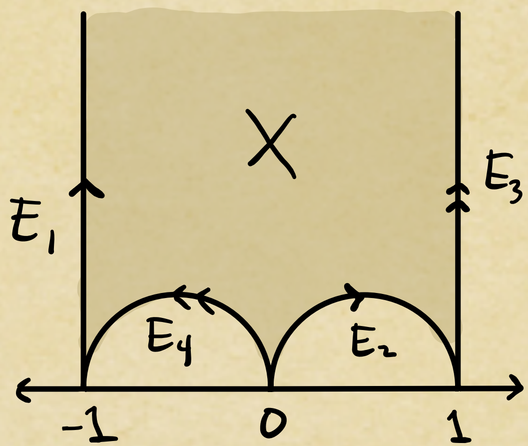


An incomplete (hyperbolic) punctured torus

Recall that we've obtained an incomplete Euclidean punctured torus by puncturing the flat torus, and we can build a complete hyperbolic punctured torus as a quotient of a hyperbolic square.

But completeness is fickle.

Consider the polygon $X \subset \mathbb{H}^2$ we used to build the punctured torus. The edge decorations tell us that we have



$$\varphi_1(z) = \frac{z+1}{z+a} \quad ; \quad \varphi_3(z) = \frac{z-1}{-z+b}, \quad \text{for some } a, b > 1.$$

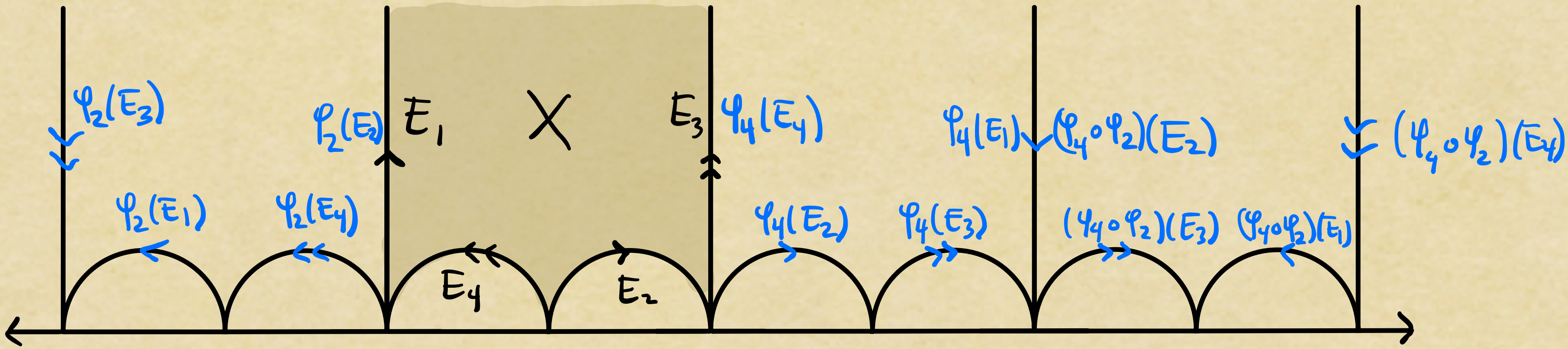
Let's consider the case $a \neq b$, which is what we'd expect generically, and attempt to tessellate $\mathbb{H}^2$. WLOG, $a > b$.

We can start with $\hat{X} = X \cup \varphi_2(X) \cup \varphi_4(X) \cup (\varphi_4 \circ \varphi_2)(X)$.

An incomplete (hyperbolic) punctured torus

$\hat{X} = X \cup \varphi_2(X) \cup \varphi_4(X) \cup (\varphi_4 \circ \varphi_2)(X)$

(Not to scale.)



We can glue the leftmost edge $\varphi_2(E_3)$ of $\hat{X}$ to the rightmost edge $(\varphi_4 \circ \varphi_2)(E_4)$ via $\varphi_4 \circ \varphi_2 \circ \varphi_3 \circ \varphi_1$.

Check:

$$(\varphi_4 \circ \varphi_2 \circ \varphi_3 \circ \varphi_1)(z) = \left(\frac{b-1}{a-1}\right)^2 z + \frac{(ab-1)(a+b-2)}{(a-1)^2}$$

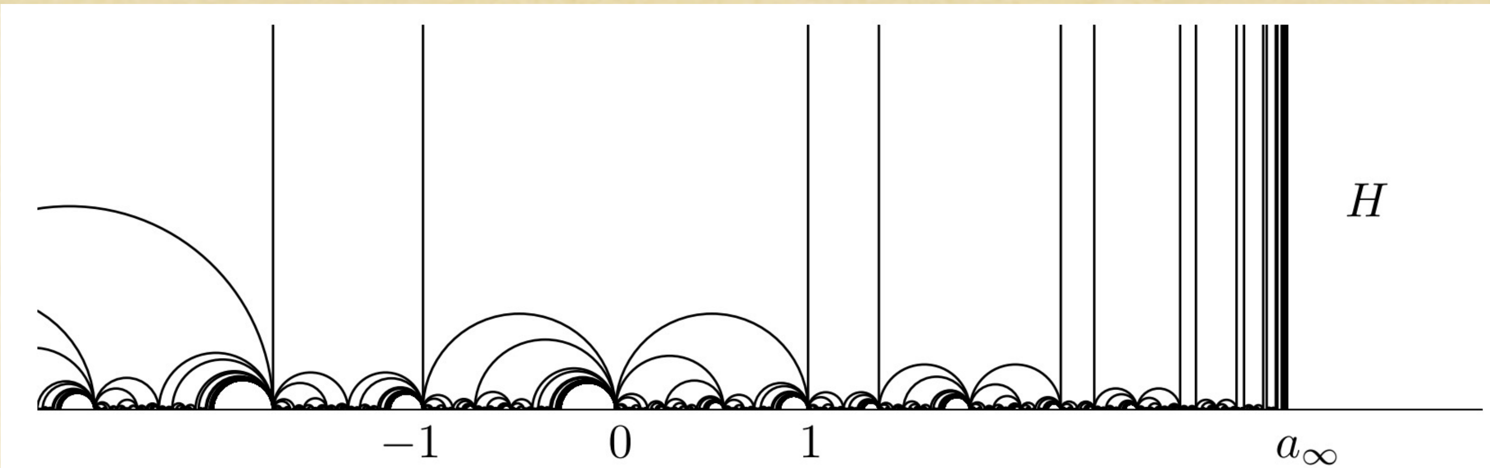
Then $(\varphi_4 \circ \varphi_2 \circ \varphi_3 \circ \varphi_1)^n(0) = \sum_{k=0}^{n-1} \left(\frac{b-1}{a-1}\right)^{2k} \frac{(ab-1)(a+b-2)}{(a-1)^2}$

An incomplete (hyperbolic) punctured torus

Then $(\varphi_4 \circ \varphi_2 \circ \varphi_3 \circ \varphi_1)^n(0) = \sum_{k=0}^{n-1} \underbrace{\left(\frac{b-1}{a-1}\right)}_{\sqrt{r}}^{2k} \underbrace{\frac{(ab-1)(a+b-2)}{(a-1)^2}}_{a_0}$

Since $a > b$, this series will converge to $a_\infty = \dfrac{ab-1}{a-b} = \dfrac{a_0}{1-r}$.

Upshot:



So $(\overline{X}, \overline{d})$ is not complete. (If it were, we'd get a tessellation of $\mathbb{H}^2$.) The same argument works for $a < b$.