Math 4803 March 25, 2024

LAST TIME

Bounded polygons in $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, & (S^2, d_{sph}) are compact, and thus their quotients by edge gluings are complete.

Upshot: Edge gluings of bounded polygons which satisfy the angle conditions give tessellations.

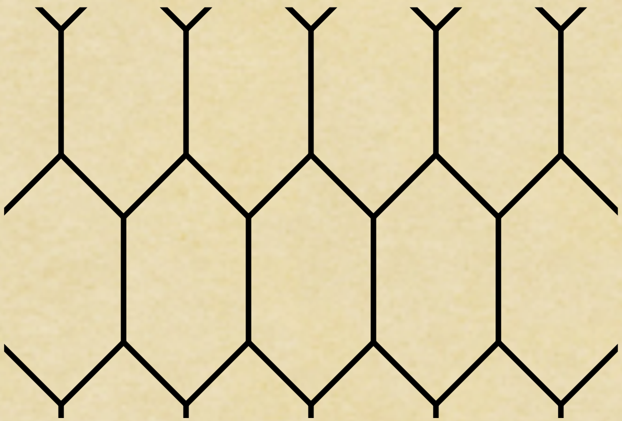
TODAY

- Precluding some tessellations of $(\mathbb{R}^2, d_{euc})$.
- An example in $(\mathbb{H}^2, d_{hyp})$.
- Tessellations by triangles.
- A tessellation by an unbounded polygon.

Convex tilings of $(\mathbb{R}^2, d_{euc})$

Fact. There is no tessellation of $(\mathbb{R}^2, d_{euc})$ whose tiles are convex polygons with more than 6 sides.

(Fake proof / reason it's true)



A tessellation of $(\mathbb{R}^2, d_{euc})$ gives us a planar graph (kind of), and these satisfy Euler's formula:

$$V - E + F = 2.$$

Each edge is shared by 2 faces, so $E = \frac{1}{2}nF$, where each face has n edges. Convexity ensures that each vertex is shared by at least 3 faces, so $V \leq \frac{1}{3}nF$.

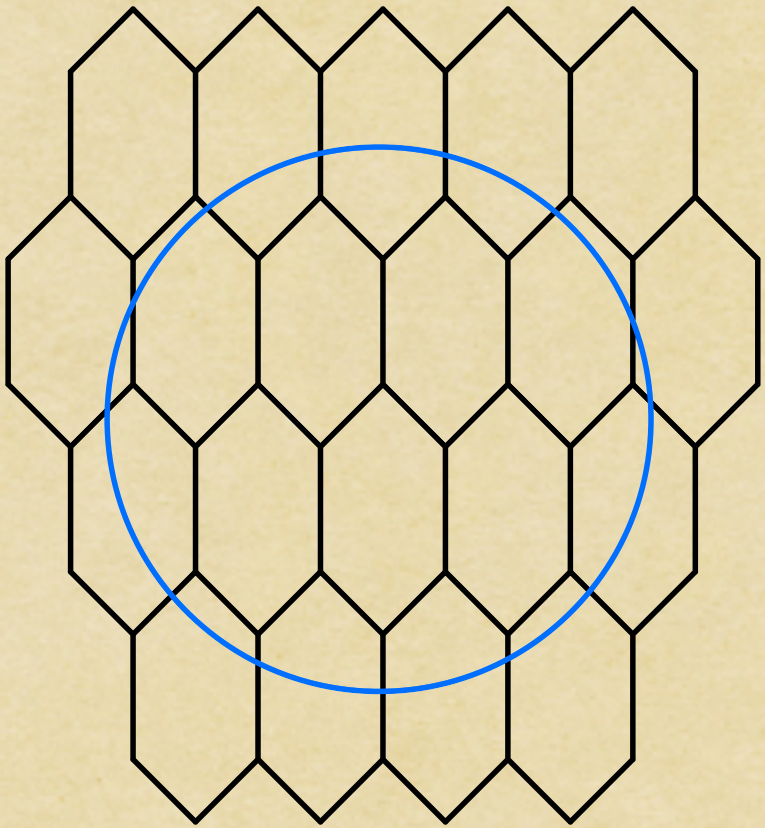
Then $2 = V - E + F = V - \frac{1}{2}nF + F$

$$\leq \frac{1}{3}nF - \frac{1}{2}nF + F = \left(\frac{2n - 3n + 6}{6}\right)F = \left(\frac{6-n}{6}\right)F$$

If $n \geq 7$, then the RHS is negative. $\times$ $\diamond$

Convex tilings of $(\mathbb{R}^2, d_{euc})$

Fact. There is no tessellation of $(\mathbb{R}^2, d_{euc})$ whose tiles are convex polygons with more than 6 sides.



Problem: We don't really have a planar graph.

Fix: For $R>0$, just think about the faces intersecting $B_{d_{euc}}(p_0, R)$

Then $F \sim R^2$

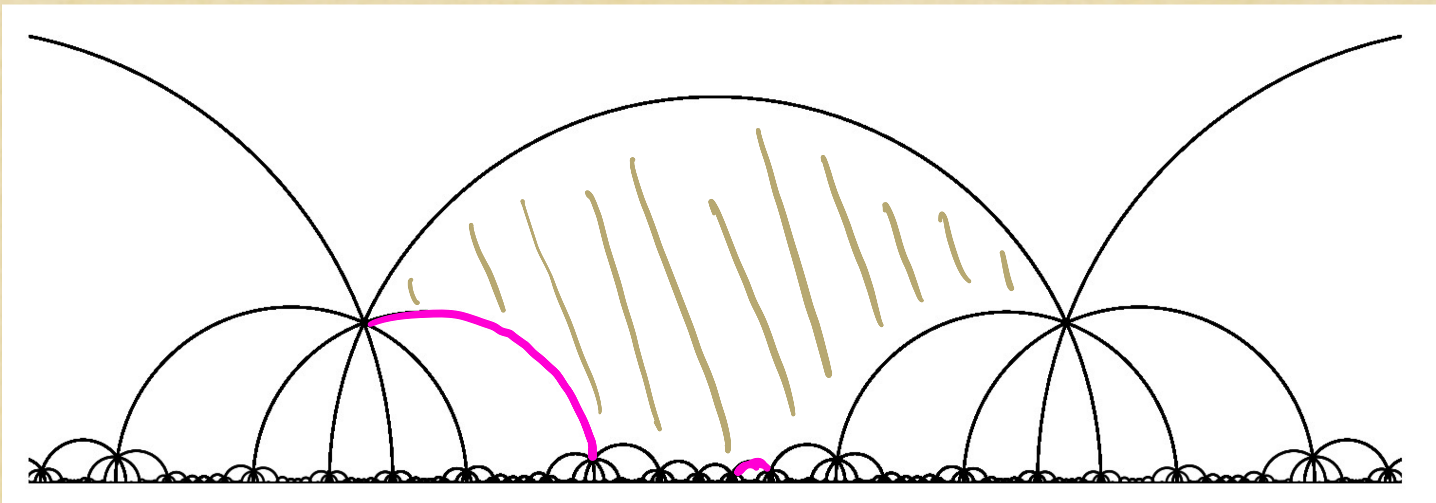
Comparing E & V to F gets more annoying, but it works out.

$F_\partial \sim R.$

Tessallation(s) of $(\mathbb{H}^2, d_{hyp})$ by bounded octagons

Unsurprisingly, $(\mathbb{H}^2, d_{hyp})$ admits some tessellations which are prohibited in $(\mathbb{R}^2, d_{euc})$. For example, we built an octagon $X \subset \mathbb{H}^2$ which admits an edge gluing by translations

We can use this to tessellate $(\mathbb{H}^2, d_{hyp})$:



$F_I \sim 4\pi \sinh^2(R/2)$

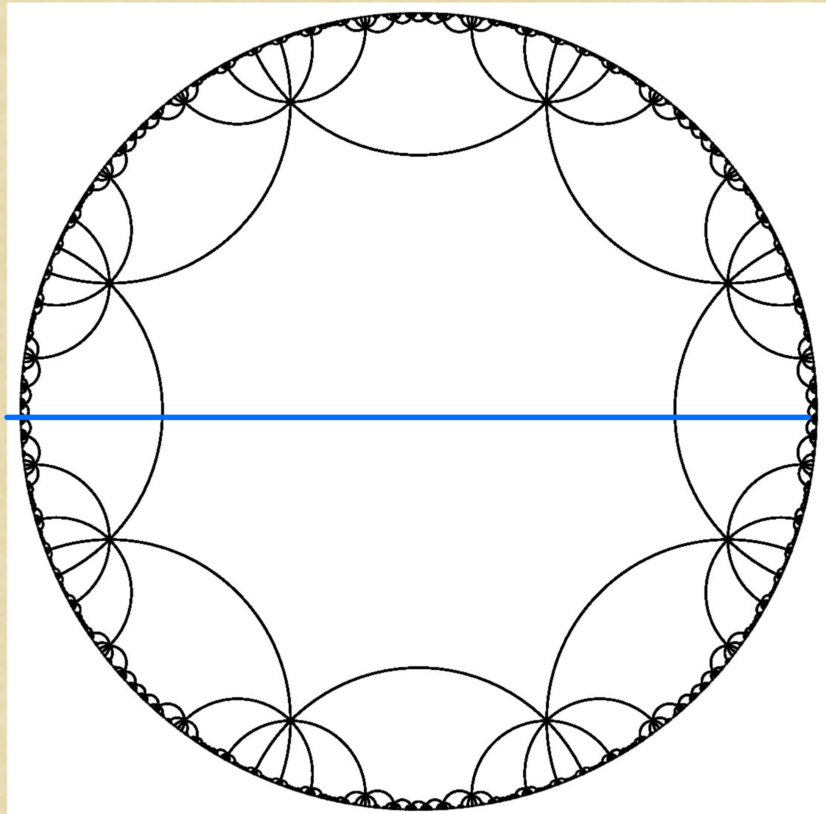
$F_\partial \sim 2\pi \sinh(R)$

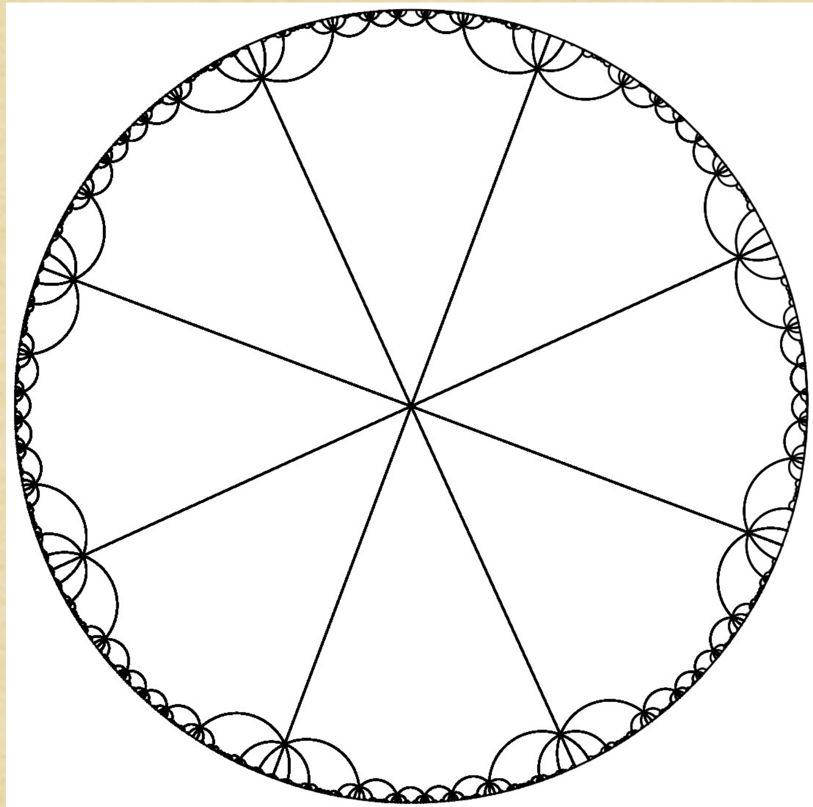
Why doesn't our Euler's formula argument work?

$F_I \sim F_\partial$, so we can't ignore effect of frontier tiles

Tessallation(s) of $(\mathbb{H}^2, d_{hyp})$ by bounded octagons

This one looks even cooler in the disc model:





By tessellating $(\mathbb{H}^2, d_{hyp})$ with lots of different triangles (coming up), we can construct tessellations by a variety of polygons.

Tessellations by triangles

Thm. For any integers $a,b,c \geq 2$,

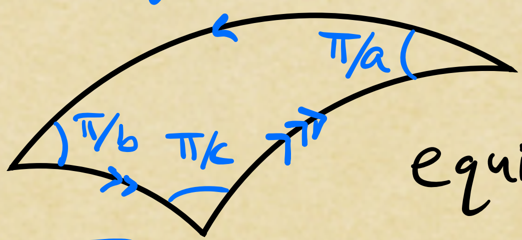
(1) if $\frac{\pi}{a}+\frac{\pi}{b}+\frac{\pi}{c}=\pi$, there is a tessellation of $(\mathbb{R}^2, d_{euc})$ by Euclidean triangles of angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$;

(2) if $\frac{\pi}{a}+\frac{\pi}{b}+\frac{\pi}{c}<\pi$, there is a tessellation of $(\mathbb{H}^2, d_{hyp})$ by hyperbolic triangles of angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$;

(3) if $\frac{\pi}{a}+\frac{\pi}{b}+\frac{\pi}{c}>\pi$, there is a tessellation of (S^2, d_{sph}) by spherical triangles of angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$.

(Proof.) Prop 5.13 $\Rightarrow$ the triangle we want exists.

For gluing data we take



. Then the vertices fall into 3 equivalence classes, and their angle sums are $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$. Since T is a bounded polygon, our previous theorems give a tessellation. $\diamond$

Rmk. Our previous proof in $(\mathbb{R}^2, d_{euc})$ used a different gluing.

Tessellations by triangles

How many tessellations does this give us?

Euclidean: $\left\{ \frac{\pi}{a} + \frac{\pi}{b} + \frac{\pi}{c} = \pi \right.$, with $a \le b \le c$.

Then $\frac{\pi}{a} \geq \frac{\pi}{3} \longrightarrow a = 2$ OR $a = 3$

If $a = 2$, then $\frac{\pi}{b} + \frac{\pi}{c} = \frac{\pi}{2} \rightarrow \frac{\pi}{b} \geq \frac{\pi}{4} \rightarrow b = 3$ OR $b = 4$

If $b = 3$, then $\frac{\pi}{c} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \rightarrow \boxed{\{2, 3, 6\}}$

If $b = 4$, then $\frac{\pi}{c} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \rightarrow \boxed{\{2, 4, 4\}}$

If $a = 3$, then $\frac{\pi}{b} + \frac{\pi}{c} = \frac{2\pi}{3} \Longrightarrow b = c = 3$

$\boxed{\{3, 3, 3\}}$

Tessellations by triangles

How many tessellations does this give us?

Euclidean: $\{a,b,c\} = \{2,3,6\}, \{2,4,4\}$, or $\{3,3,3\}$.

Spherical: $\oint \frac{\pi}{a} + \frac{\pi}{b} + \frac{\pi}{c} > \pi$, with $a \leq b \leq c$.

Thus

Then $a < 3$. ~~If~~ $a = 2$, then

$$\frac{\pi}{b} + \frac{\pi}{c} > \frac{\pi}{2} \Rightarrow b < 4$$

If $b=2$, then $\frac{\pi}{c} > 0 \longrightarrow \{2,2,c\}$

If $b=3$, then $\frac{\pi}{c} > \frac{\pi}{6} \Rightarrow c < 6$

This gives $\{2,3,3\}, \{2,3,4\}, \{2,3,5\}$

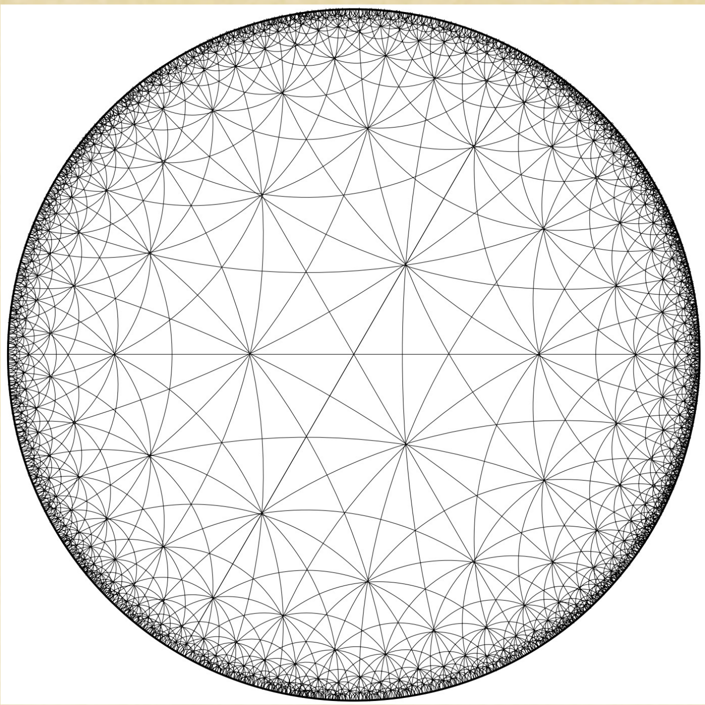
Tessellations by triangles

How many tessellations does this give us?

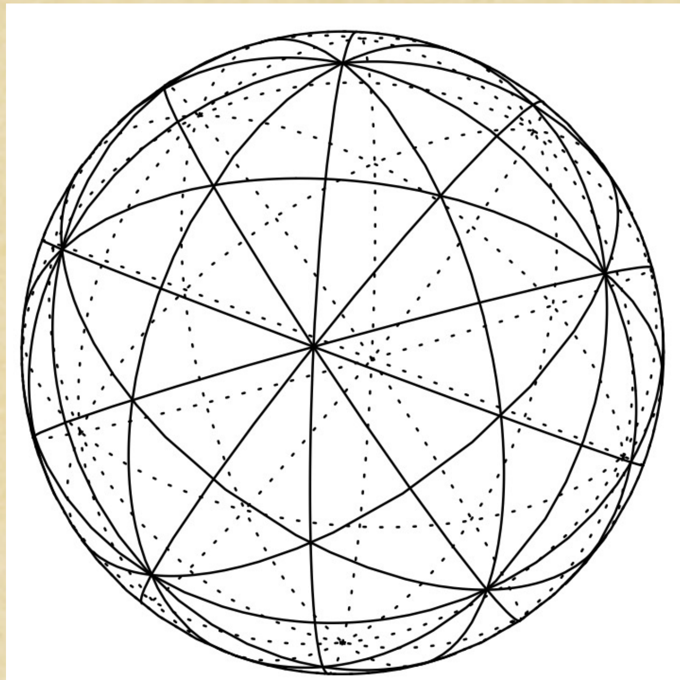
Euclidean: $\{a,b,c\} = \{2,3,6\}, \{2,4,4\}$, or $\{3,3,3\}$.

Spherical: $\{a,b,c\} = \{2,3,3\}, \{2,3,4\}, \{2,3,5\}$, or $\{2,2,c\}$.

Hyperbolic: all other options



$\{2,3,7\}$



$\{2,3,5\}$

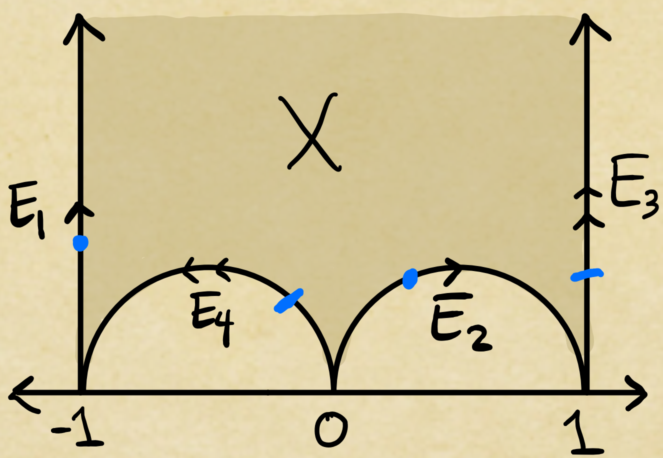
A tessellation of $(\mathbb{H}^2, d_{hyp})$ by an unbounded square

All of today's tessellations use the following:

> <u>Prop.</u> Let X be a bounded polygon in $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}). Then any edge gluing of X which <u>satisfies the angle condition on vertices</u> leads to a tessellation.

This works because edge gluings on bounded polygons lead to <u>complete</u> quotient spaces. Now we'll see an example which uses an unbounded polygon.

Consider the following edge gluing of a square:



To apply the <u>tessellation theorem</u>, we need to verify that $(\overline{X}, \overline{d})$ is <u>complete</u>.

A tessellation of $(\mathbb{H}^2, d_{hyp})$ by an unbounded square

Recall that we can decompose X as

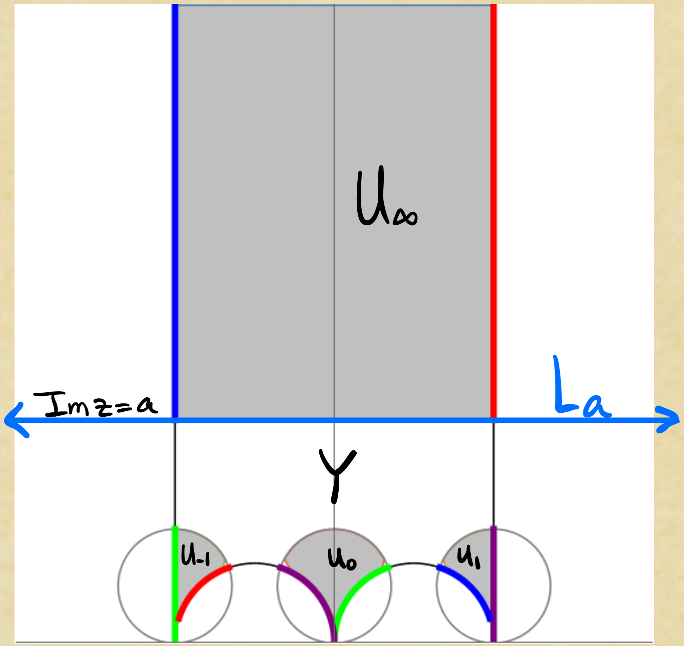
$$X = U \cup Y,$$

with Y a bounded "polygon" and

$$U = U_\infty \cup U_{-1} \cup U_0 \cup U_1,$$

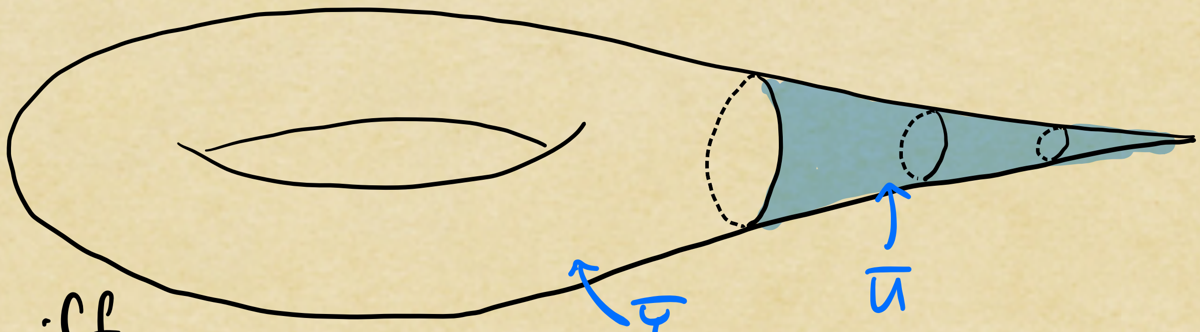
where, for instance,

$$U_\infty = \{z \in X \mid \operatorname{Im} z \geq a\}.$$



This gives a decomposition

$$\overline{X} = \overline{U} \cup \overline{Y},$$



and $\overline{X}$ is complete iff $\overline{U}$ and $\overline{Y}$ are complete. Note that Y a bounded "polygon"

$\Rightarrow \overline{Y}$ is complete. It remains to check $\overline{U}$.

A tessellation of $(\mathbb{H}^2, d_{hyp})$ by an unbounded square

Claim. $\overline{U}$ is complete.

First, we know that $\overline{U}$ is isometric to the pseudosphere, $(S_a, d_{S_a}) \subset (\mathbb{R}^3, d_{euc})$. Because $(\mathbb{R}^3, d_{euc})$ is complete, any finite-length sequence in (S_a, d_{S_a}) will converge in $(\mathbb{R}^3, d_{euc})$. But S_a is determined by the equations and inequalities

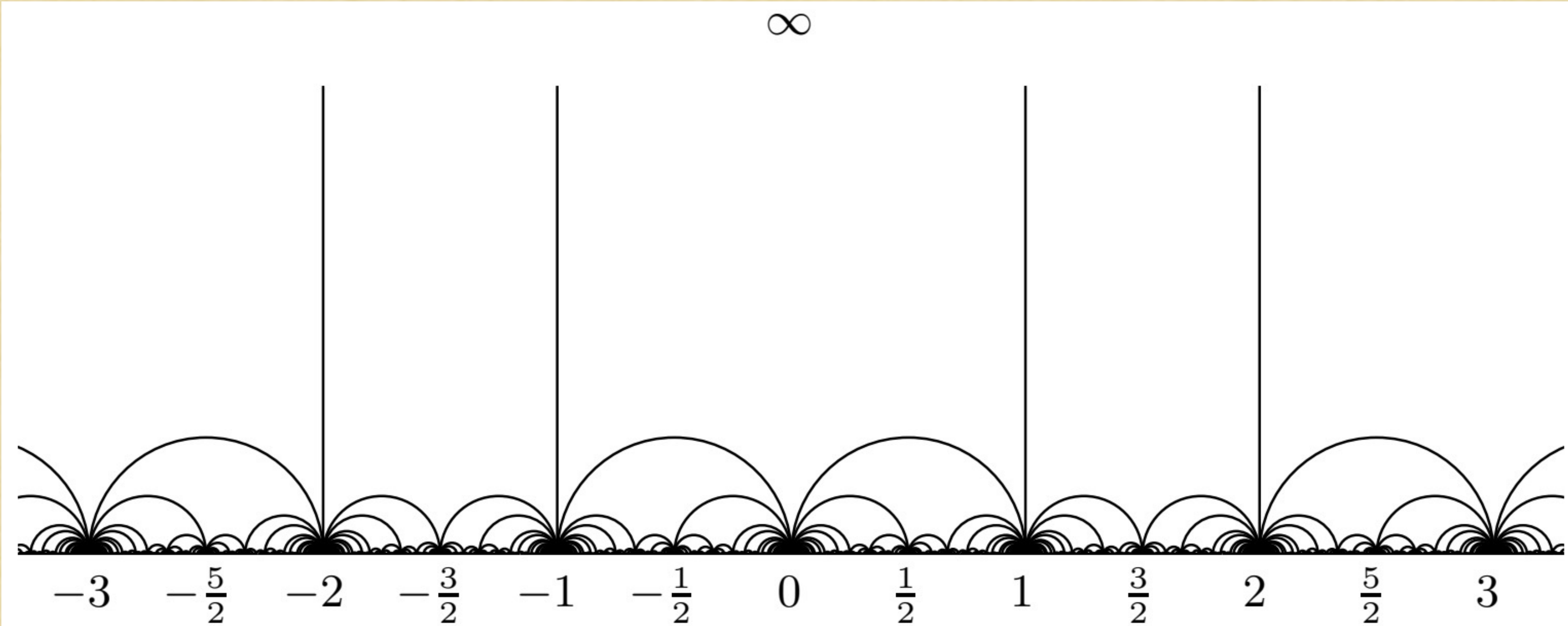
$$x = t - \tanh t, \quad y = \operatorname{sech} t \cdot \cos s, \quad z = \operatorname{sech} t \cdot \sin s,$$
$$\cosh^{-1}(a\pi/3) \leq t < \infty, \quad 0 \leq s \leq 2\pi,$$

No strict inequalities

and thus the limit point will be contained in S_a.

So any finite-length sequence in (S_a, d_{S_a}) will converge in (S_a, d_{S_a}). S_a is complete $\Rightarrow$ $\overline{U}$ is complete $\Rightarrow$ $\overline{X}$ is complete $\Rightarrow$ we get a tessellation

A tessellation of $(\mathbb{H}^2, d_{hyp})$ by an unbounded square



The edge gluing is not by translation!