Math 4803 March 13, 2024

LAST TIME

Generalities about completeness

& compactness.

TODAY

A bit more on compactness, then tessellations by bounded polygons.

Compactness in the fundamental geometries

Here's some vocab that will help us characterize compact subsets of some metric spaces.

Consider a subset Y of a metric space (X,d).

We call Y **bounded** if $\exists P_0 \in X$ and $R > 0$ s.t. $Y \subseteq B_d(P_0, R)$.

Bounded or not, Y partitions X into three types of points: $P \in X$ is

- an interior point of Y if $\exists\, \varepsilon > 0$ s.t. $B_d(P, \varepsilon) \subseteq Y$;
- an exterior point of Y if $\exists\, \varepsilon > 0$ s.t. $B_d(P, \varepsilon) \cap Y = \emptyset$;
- a boundary point of Y if it's neither of these.

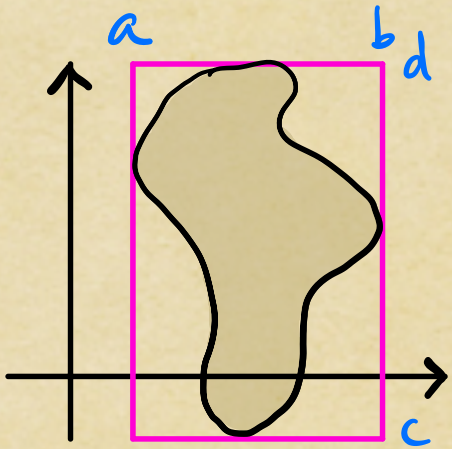
We call Y **closed** if P a boundary point $\Rightarrow P \in Y$.

Compactness in $(\mathbb{R}^2, d_{euc})$

Thm Every closed, bounded subset of $(\mathbb{R}^2, d_{euc})$ is compact.

Rmk: This is actually an iff, but we'll prove only what we need.

(Proof.) We'll take for granted that every closed interval $[a,b]$ in $(\mathbb{R}, |x-y|)$ is compact (c.f. Bolzano–Weierstrass).



Now consider a sequence $(P_n)_{n\in\mathbb{N}}$ in a closed, bounded subset X of $(\mathbb{R}^2, d_{euc})$. Since X is bounded, $X \subseteq [a,b]\times[c,d]$, for some $a<b$, $c<d$ in $\mathbb{R}$. This allows us to find a subsequence $(P_{n_k})_{k\in\mathbb{N}}$ s.t. $(x_{n_k})_{k\in\mathbb{N}}$ converges to some x_∞.

But the y-values are also bounded, so $\exists$ subsubsequence $(P_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ s.t. $(y_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ converges to y_∞.

Compactness in $(\mathbb{R}^2, d_{euc})$

<u>Thm</u> Every closed, bounded subset of $(\mathbb{R}^2, d_{euc})$ is compact.

(Proof, cont'd) We now have $(P_{n_{k_i}})_{i\in\mathbb{N}}$ s.t.

$$(x_{n_{k_i}})_{i\in\mathbb{N}} \to x_\infty \;\;\&\;\; (y_{n_{k_i}}) \to y_\infty ,$$

so $(P_{n_{k_i}})_{i\in\mathbb{N}} \longrightarrow P_\infty = (x_\infty, y_\infty)$. It remains to check <u>$P_\infty \in X$</u>. But any ball $B_{d_{euc}}(P_\infty, \varepsilon)$ <u>contains</u> <u>elements of X</u>, so P_∞ cannot be <u>exterior to X</u>.

specifically, subsubseq. points

Since X <u>is closed</u>, $P_\infty \in X$.

So every sequence in X admits a subsequence which converges in X. i.e., X is compact. $\diamond$

Compactness in $(\mathbb{H}^2, d_{hyp})$

Thm Every closed, bounded subset of $(\mathbb{H}^2, d_{hyp})$ is compact.

(Proof.) § $X \subset \mathbb{H}^2$ is closed & bounded. Then $\exists\, P_0 \in \mathbb{H}^2$ & $K \gg 0$ s.t. $X \subsetneq B_{d_{hyp}}(P_0, K)$. Now if $(P_n)_{n\in\mathbb{N}}$ is a sequence in X, then $d_{hyp}(P_n, P_0) < K$, $\forall\, n \in \mathbb{N}$, so $y_0 e^{-K} < y_n < y_0 e^{K}$ (double inequality), where $P_0 = (x_0, y_0)$ and $P_n = (x_n, y_n)$. As in the completeness proof, we find that $d_{euc}(P_n, P_0) \leq C_2\, d_{hyp}(P_n, P_0)$, where $C_2 = y_0 e^{K}$. Thus $(P_n)_{n\in\mathbb{N}}$ is contained in the compact in d_{euc} set $B_{d_{euc}}(P_0, C_2 K)$, and thus admits a subsequence converging to P_∞. Because $d_{hyp}(P_n, P_\infty) \leq \frac{1}{C_1} d_{euc}(P_n, P_\infty)$, the sequence $(P_{n_k})_{k\in\mathbb{N}}$ converges to P_∞ w.r.t. d_{hyp} as well. (We know that $P_\infty \in \mathbb{H}^2$ b/c $y_n > y_0 e^{-K} > 0$)

p.s. $P_\infty \in X$, since X is closed

$\diamond$

Compactness in (S^2, d_{sph})

Thm Every closed subset of (S^2, d_{sph}) is compact.

(Proof.) Let X be a closed subset of (S^2, d_{sph}). Then X is a closed & bounded subset of $(\mathbb{R}^3, d_{euc})$. By the 3D version of our Euclidean statement, X is compact as a subset of $(\mathbb{R}^3, d_{euc})$. So any sequence $(P_n)_{n\in\mathbb{N}}$ in X admits a subsequence $(P_{n_k})_{k\in\mathbb{N}}$ which converges w.r.t. d_{euc} to P_∞. One can show that

$$d_{sph}(P_{n_k}, P_\infty) = 2\arcsin\left(\frac{d_{euc}(P_{n_k}, P_\infty)}{2}\right).$$

Since $d_{euc}(P_{n_k}, P_\infty) \to 0$ as $k \to \infty$, we have that $d_{sph}(P_{n_k}, P_\infty) \to 0$ as $k\to\infty$, so $(P_{n_k})_{k\in\mathbb{N}} \to P_\infty$ in d_{sph}. $\diamond$

Tessellations by bounded polygons

All these generalities about compactness buy us an important fact:

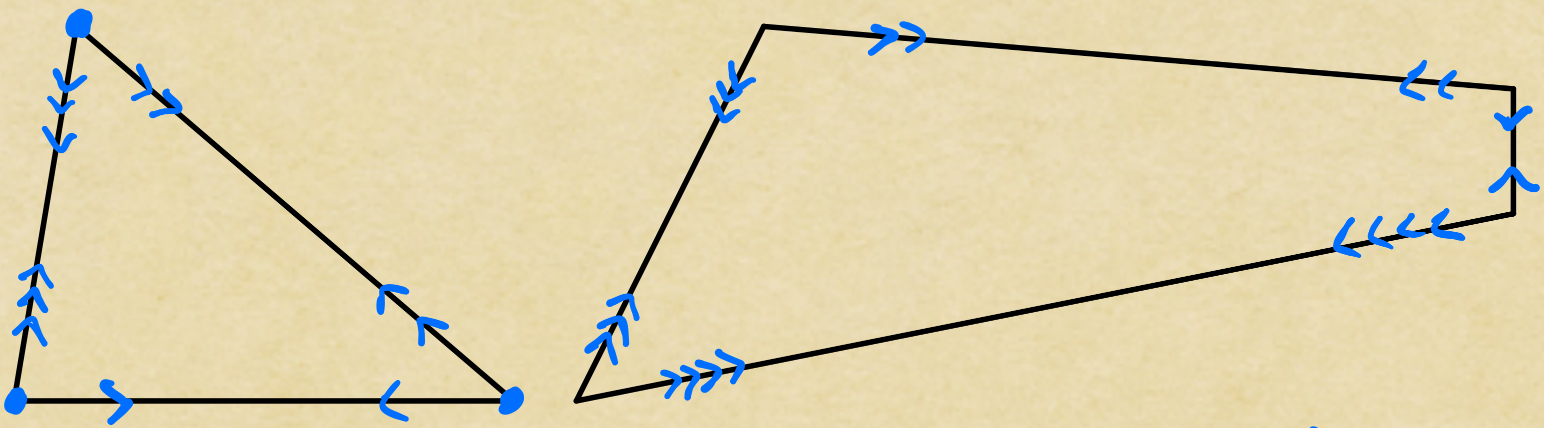
<u>Prop.</u> Let X be a bounded polygon in $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}). Then any edge gluing of X which satisfies the angle condition on vertices leads to a tessellation.

(Proof.) According to the tessellation theorem, we'll win as long as the quotient $(\overline{X}, \overline{d})$ is complete. By definition, all polygons are closed; since X is also bounded, X is compact. Now the quotient map $\pi : X \to \overline{X}$ is continuous, so $\pi(X) = \overline{X}$ is compact, and therefore complete. So the tessellation theorem gives a tessellation. $\diamond$

Tessellations by bounded polygons in $(\mathbb{R}^2, d_{euc})$

<u>Prop</u>. Given any triangle or quadrilateral X in $(\mathbb{R}^2, d_{euc})$, there is a tessellation of $(\mathbb{R}^2, d_{euc})$ whose tiles are all isometric to X.

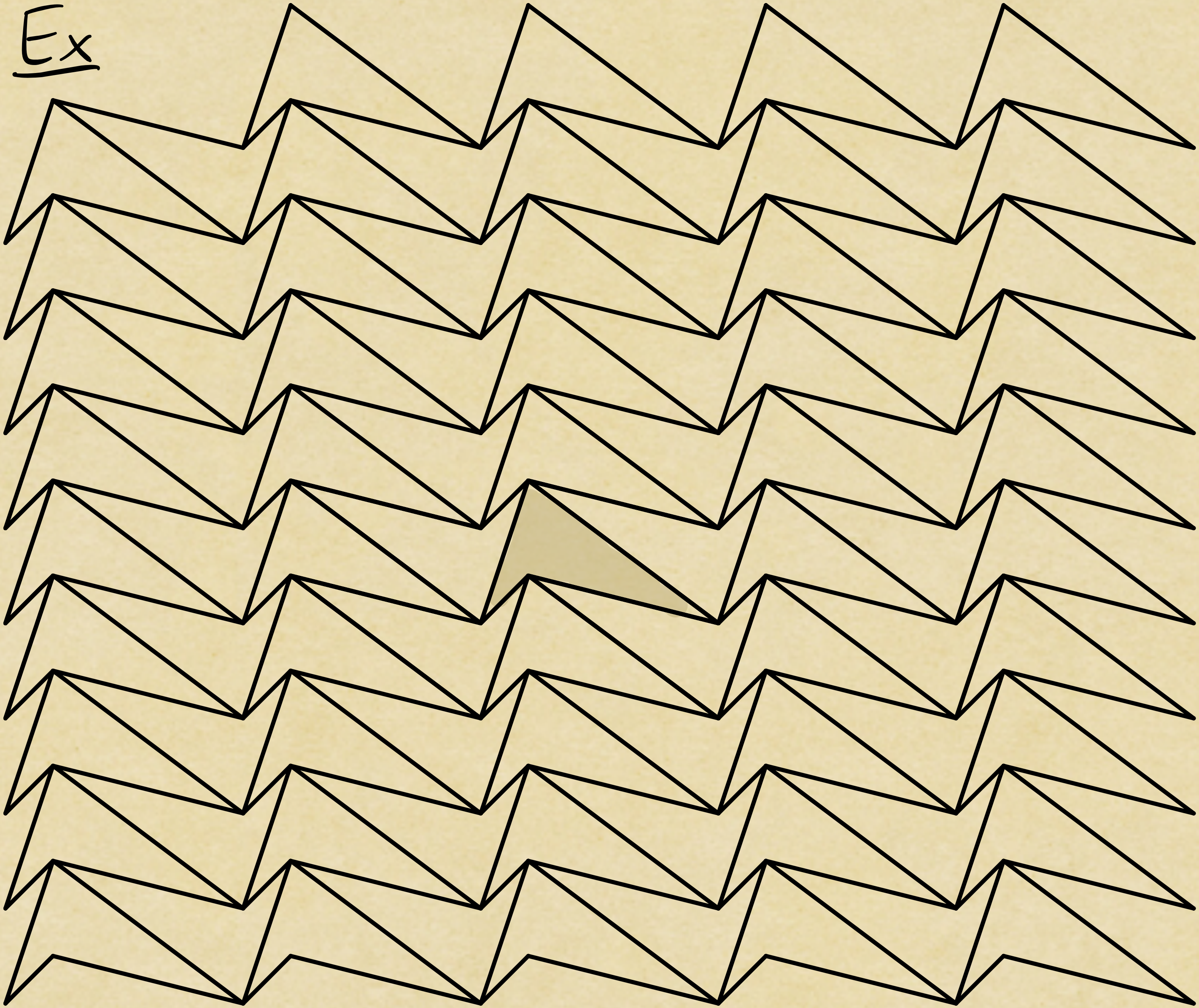
(Proof.) All we need to do is construct an edge gluing which satisfies the angle condition.



We can glue each edge to itself via a reflection across its midpoint. Triangles and quadrilaterals have angle sum π & 2π, respectively. ◇

Tessellations by bounded polygons in $(\mathbb{R}^2, d_{euc})$

Ex



Tessellations by bounded polygons in $(\mathbb{R}^2, d_{euc})$

Facts

- No convex polygon with more than six sides can tile the plane.
- There are exactly 3 types of convex hexagons which tile the plane.
- In 1918, it was conjectured that there are exactly 5 types of convex pentagons which tile the plane. In 2015, a 15^{th} type was found.