Math 4803 March 11, 2024

LAST TIME

We proved the tessellation theorem, for which completeness of the quotient $(\bar{X}, \bar{d})$ was crucial.

TODAY

We'll investigate completeness and compactness more closely.

Did we have to assume completeness?

The tessellation theorem

Thm. Let X be a connected polygon in the Euclidean plane, hyperbolic plane, or sphere, and suppose that an edge gluing $\{\varphi_i : E_i \to E_{i\pm 1}\}$ has been specified. If

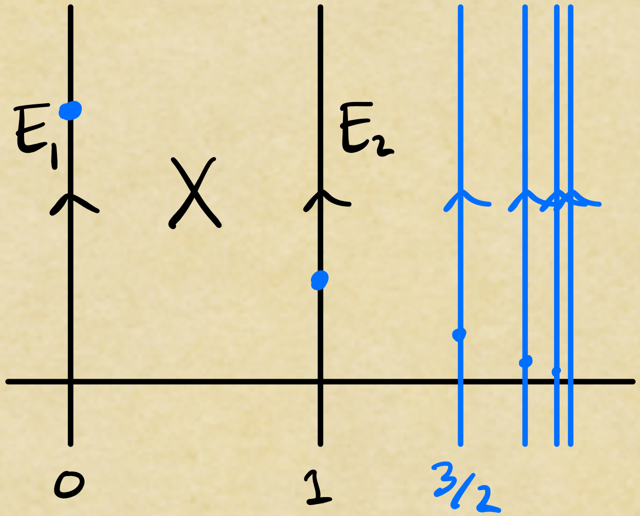
(1) for every vertex $P \in X$, $\sum_{Q \sim P} \measuredangle(Q) = \frac{2\pi}{n}$, where $n > 0$ is an integer which may depend on P;

(2) the quotient metric space $(\overline{X}, \overline{d}_X)$ is complete;

then the family $\{\varphi(X) \mid \varphi \in \Gamma\}$ is a tessellation of the Euclidean plane, hyperbolic plane, or sphere.

i.e., does every edge gluing which satisfies (1) automatically give a tessellation?

Did we have to assume completeness?



Consider the polygon

$$X = \{ z \in \mathbb{H}^2 \mid 0 \le \operatorname{Re} z \le 1 \}$$

and the edge gluing

$$\varphi_1 : E_1 = \{\operatorname{Re} z = 0\} \to E_2 = \{\operatorname{Re} z = 1\}$$

$$z \longmapsto \tfrac{1}{2} z + 1$$

Notice that $\varphi(i) = \frac{1}{2} i + 1$

$$\varphi^2(i) = \tfrac{1}{2}\left(\tfrac{1}{2} i + 1\right) + 1 = \tfrac{1}{4} i + \tfrac{1}{2} + 1$$

$$\varphi^3(i) = \tfrac{1}{8} i + \tfrac{1}{4} + \tfrac{1}{2} + 1$$

$$\varphi^n(i) = \frac{1}{2^n} i + \sum_{k=0}^{n-1} \left(\tfrac{1}{2}\right)^k$$

So $\lim_{n\to\infty} \varphi^n(E_1) = \left\{ \operatorname{Re} z = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \right\} = \{ \operatorname{Re} z = 2 \}$.

This edge gluing does not cover all of $\mathbb{H}^2$.

Did we have to assume completeness?

Our assumption of completeness was crucial for ensuring that our tiling process covered the entire space.

We'll now address some generalities about completeness.

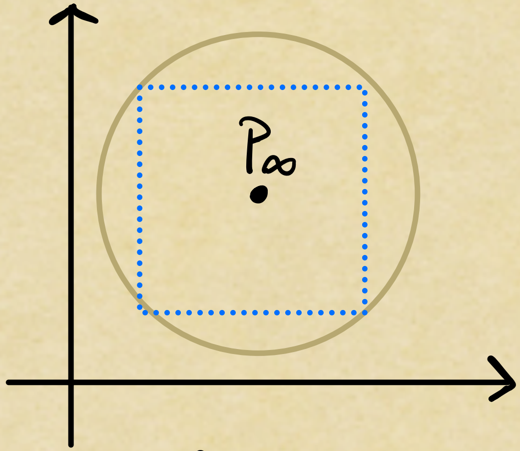
Recall that (X,d) is complete if every finite-length sequence converges.

We say that $(p_n)_{n\in\mathbb{N}}$ converges to p_∞ if, for every $\varepsilon>0$, $\exists\, n_0 \geq 1$ s.t. $p_n \in B_d(p_\infty, \varepsilon)$, for every $n \geq n_0$.

$(\mathbb{R}^2, d_{euc})$ is complete

Thm. The Euclidean plane $(\mathbb{R}^2, d_{euc})$ is complete.

(Proof.) This is pretty easy once we give ourselves:



Fact: $\mathbb{R}$ is complete under $d(x,y) = |x-y|$

Now suppose $(P_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^2$ with finite length and write

$$P_n = (x_n, y_n), \quad \forall\, n \geq 1.$$

Then $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ with length

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) = \sum_{n=1}^{\infty} |x_n - x_{n+1}| = \sum_{n=1}^{\infty} \sqrt{(x_n - x_{n+1})^2} \leq \sum_{n=1}^{\infty} d_{euc}(P_n, P_{n+1}) < \infty.$$

So $(x_n)_{n\in\mathbb{N}}$ must converge, and the same is true of $(y_n)_{n\in\mathbb{N}}$. One can check that this implies the convergence of $(P_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$.

$\diamond$

$(\mathbb{H}^2, d_{hyp})$ is complete

Thm. The hyperbolic plane $(\mathbb{H}^2, d_{hyp})$ is complete.

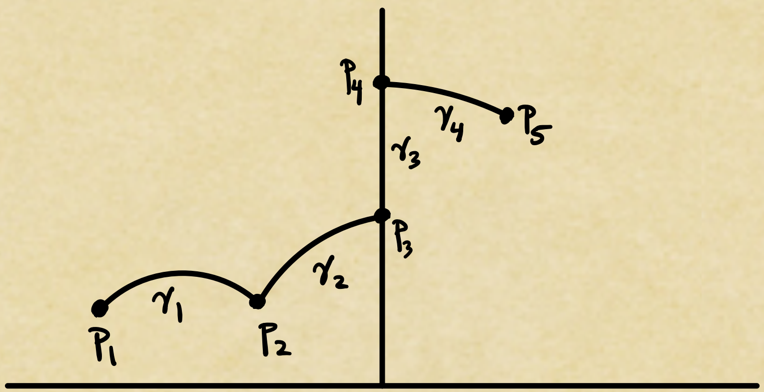
(Proof.) Our strategy is to show that if $\sum_{k=1}^{\infty} d_{hyp}(P_k, P_{k+1})$ is finite, then $\underline{\sum_{k=1}^{\infty} d_{euc}(P_k, P_{k+1})}$ is finite. Then we'll show that $\underline{\text{convergence in } (\mathbb{R}^2, d_{euc})}$ implies convergence in $(\mathbb{H}^2, d_{hyp})$.

Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of finite length L in $(\mathbb{H}^2, d_{hyp})$, and let γ_n denote ____________________, for $n \geq 1$.

Claim. $\forall\, n \geq 1$ and $\forall\, P \in \gamma_n$,

$$y_1 e^{-L} < y < y_1 e^{L},$$

where $P = (x, y)$ & $P_n = (x_n, y_n)$.



$(\mathbb{H}^2, d_{hyp})$ is complete

Claim. $\forall\, n \geq 1$ and $\forall\, P \in \gamma_n$, $y_1 e^{-L} < y < y_1 e^{L}$.

(Proof of claim.) First, the triangle inequality gives

$$d_{hyp}(P, P_1) \leq d_{hyp}(P, P_n) + \sum_{k=1}^{n-1} d_{hyp}(P_k, P_{k+1})$$
$$\leq d_{hyp}(P_{n+1}, P_n) + \sum_{k=1}^{n-1} d_{hyp}(P_k, P_{k+1}) \leq L$$

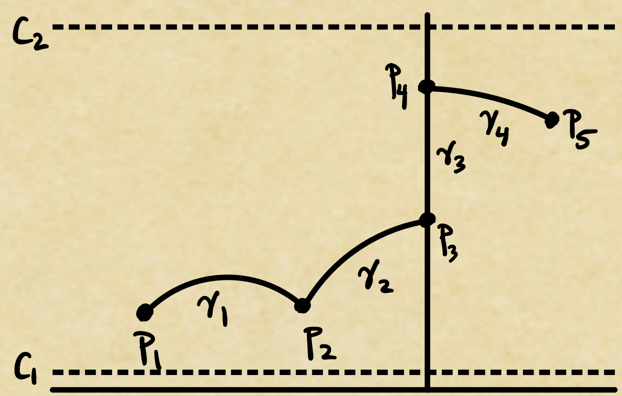
OTOH, we showed ages ago that $d_{hyp}(P, P_1) \geq \left| \ln \frac{y}{y_1} \right|$.

$\therefore \left| \ln \frac{y}{y_1} \right| \leq L \rightarrow -L \leq \ln \frac{y}{y_1} \leq L \rightarrow e^{-L} \leq \frac{y}{y_1} \leq e^{L} \rightarrow$ Done. $\square$

Letting $c_1 = y_1 e^{-L}$ & $c_2 = y_1 e^{L}$,

$$d_{hyp}(P_n, P_{n+1}) = \ell_{hyp}(\gamma_n) \geq \frac{1}{c_2} \ell_{euc}(\gamma_n)$$
$$\geq \frac{1}{c_2} d_{euc}(P_n, P_{n+1})$$

So $\sum_{n=1}^{\infty} d_{euc}(P_n, P_{n+1}) \leq c_2 L < \infty$.



$(\mathbb{H}^2, d_{hyp})$ is complete

So $(P_n)_{n\in\mathbb{N}}$ with finite length in $(\mathbb{H}^2, d_{hyp})$ must have finite length in $(\mathbb{R}^2, d_{euc})$, and therefore converges in $(\mathbb{R}^2, d_{euc})$ to P_∞.

For any $\varepsilon > 0$, we can choose $n_0 \in \mathbb{N}$ s.t.

$$n \geq n_0 \Rightarrow d_{euc}(P_n, P_\infty) < c_1 \varepsilon.$$

But then we have

$$d_{hyp}(P_n, P_\infty) \leq \frac{1}{c_1} d_{euc}(P_n, P_\infty) < \frac{c_1 \varepsilon}{c_1} = \varepsilon \checkmark$$

So $(P_n)_{n\in\mathbb{N}}$ converges to P_∞ in $(\mathbb{H}^2, d_{hyp})$. $\diamond$

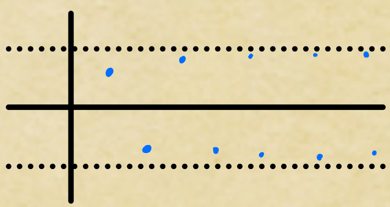
Compactness

There's another property of metric spaces which implies completeness, but is often easier to check.

Given a sequence $(p_n)_{n\in\mathbb{N}}$, a **subsequence** of $(p_n)_{n\in\mathbb{N}}$ has the form $(p_{n_k})_{k\in\mathbb{N}}$, where $n_1 < n_2 < n_3 < \cdots$ is any increasing sequence of integers.

e.g., $\left(3-\left(\frac{1}{2}\right)^{2k}\right)_{k\in\mathbb{N}}$ is a subsequence of $\left((-1)^n\left[3-\left(\frac{1}{2}\right)^n\right]\right)_{n\in\mathbb{N}}$

$3-\frac{1}{4}, 3-\frac{1}{16}, 3-\frac{1}{64}, \ldots$ $\qquad$ $-(3-\frac{1}{2}), 3-\frac{1}{4}, -(3-\frac{1}{8}), 3-\frac{1}{16}, \ldots$



We call a metric space (X,d) **compact** if every sequence in (X,d) admits a convergent subsequence i.e., from any $(p_n)_{n\in\mathbb{N}}$ we may find $(p_{n_k})_{k\in\mathbb{N}}$ which converges.

We're not claiming uniqueness!

Compactness

Prop. Every compact metric space is complete.

(Proof.) Let $(P_n)_{n\in\mathbb{N}}$ be a finite-length sequence in a compact metric space (X,d). Then $(P_n)_{n\in\mathbb{N}}$ admits a convergent subsequence $(P_{n_k})_{k\in\mathbb{N}}$. Let $P_\infty = \lim_{k\to\infty} P_{n_k}$.

We'll show that $(P_n)_{n\in\mathbb{N}}$ converges to P_∞. Take $\varepsilon>0$.

First, pick $k_0 \geq 1$ s.t. $k\geq k_0 \Rightarrow d(P_{n_k}, P_\infty) < \frac{\varepsilon}{2}$. We can also choose $n_0 \geq n_{k_0}$ s.t., for every $n\geq n_0$,

$$\sum_{i=n}^{\infty} d(P_i, P_{i+1}) < \frac{\varepsilon}{2}.$$

Then for any $m > n \geq n_0$,

$$d(P_n, P_m) \leq \sum_{i=n}^{m-1} d(P_i, P_{i+1}) \leq \sum_{i=n}^{\infty} d(P_i, P_{i+1}) < \frac{\varepsilon}{2}.$$

Given $n\geq n_0$, we can pick k s.t. $n_k > n$ and notice that

$$d(P_n, P_\infty) \leq d(P_n, P_{n_k}) + d(P_{n_k}, P_\infty) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

So $P_n \to P_\infty$ as $n\to\infty$. $\diamond$

Compactness

So compactness $\Rightarrow$ completeness (but not conversely !). This is useful because establishing compactness is often easier than working directly with the definition of completeness.

Prop. Let $\varphi:(X,d)\to(X',d')$ be a continuous map between metric spaces. If (X,d) is compact, then $(\varphi(X), d'|_{\varphi(X)})$ is compact.

(Proof.) Any sequence in $\varphi(X)$ has the form $(\varphi(p_n))_{n\in\mathbb{N}}$, where $(p_n)_{n\in\mathbb{N}}$ is a sequence in X. Because (X,d) is compact, $\exists$ a subsequence $(p_{n_k})_{k\in\mathbb{N}}$ which converges to p_∞. Then the subsequence $(\varphi(p_{n_k}))_{k\in\mathbb{N}}$ of $(\varphi(p_n))_{n\in\mathbb{N}}$ converges to $\varphi(p_\infty)$, since φ is continuous.

Next

- We'll characterize compact subsets of $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, & (S^2, d_{sph}).
- We'll use that last proposition + the tessellation theorem to construct lots more tessellations.