Math 4803 **January 8, 2024**

TODAY

1. Quick overview of the course
2. Warm up group work
3. Geometry of the Euclidean plane

Rough plan for topics

Goal (ambitious): Geometrization theorem for 3D manifolds

Jan: Euclidean, hyperbolic, & spherical geometries

Feb: Surfaces via gluing

Mar: Tessellations (i.e., pretty pictures)

Apr: Group actions & 3D geometries

Group work

1. Write each complex number as $x+iy$:

   (a) $\frac{1}{i}$ (b) $\frac{4+i}{6-3i}$ (c) i^{47}

2. Find the modulus of each complex number:

   (a) $\frac{3-i}{(6+2i)^3}$ (b) $(\sqrt{3}+i)(\sqrt{3}-i)$ (c) $\frac{i+2}{i-2}$

3. Convert each to polar form $re^{i\theta}$:

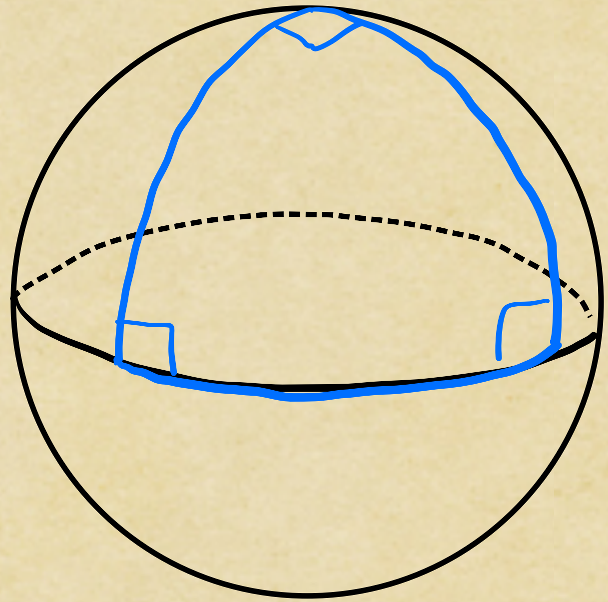
   (a) $\sqrt{3}+i$ (b) $-6+6i$

4. Convert each to rectangular form $x+iy$:

   (a) $3e^{i\pi}$ (b) $7e^{-i\pi/6}$

The Euclidean plane

We want to discuss various **geometries** in 2D.



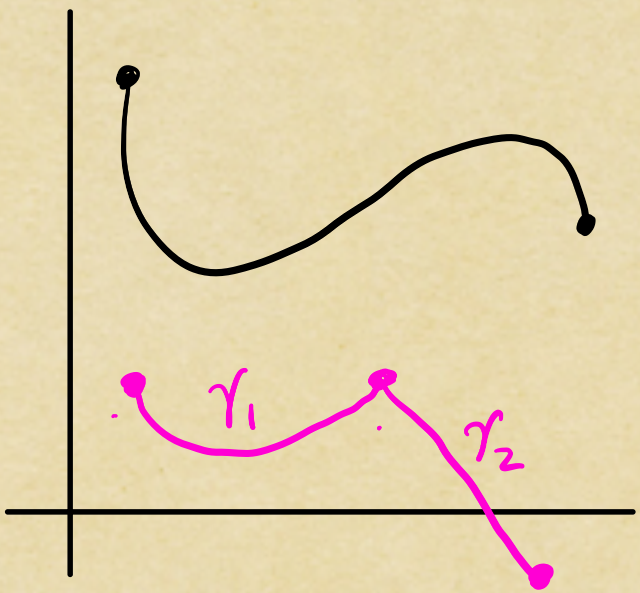
We'll start by developing the familiar geometry of the **Euclidean plane** in formal terms.

$$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}.$$

Euclidean length

A (differentiable) curve γ in $\mathbb{R}^2$ is parametrized by $[a, b] \longrightarrow \mathbb{R}^2$, a diff'able vector-valued function.

$$t \longmapsto (x(t), y(t))$$



The Euclidean length of γ is given by

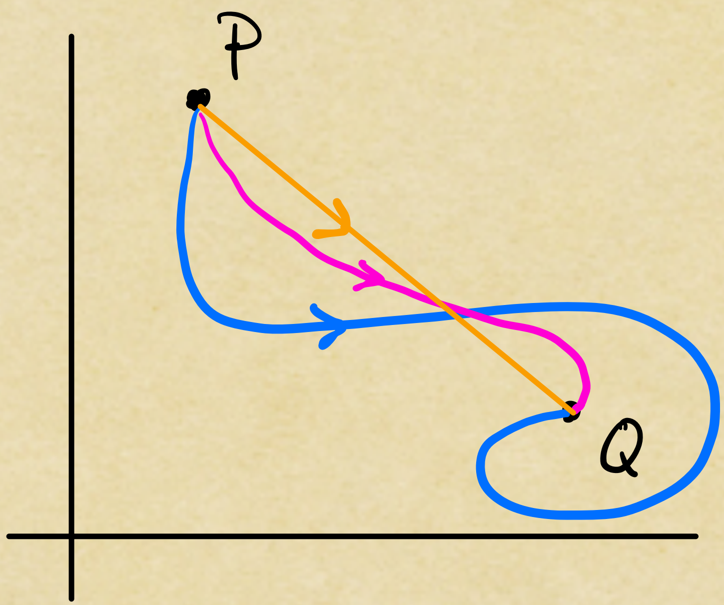
$$\ell_{euc}(\gamma) := \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

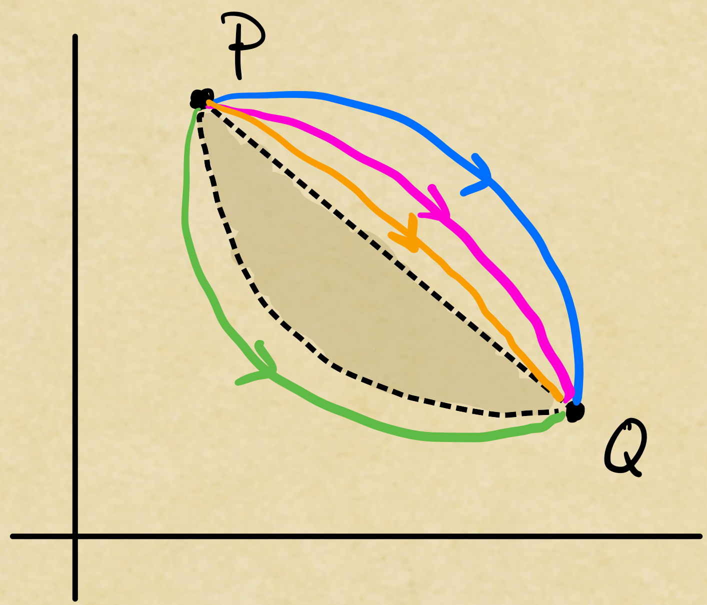
We will also allow γ to be piecewise differentiable (requires continuity) $\Rightarrow \ell_{euc}(\gamma) := \sum_{i=1}^{n} \ell_{euc}(\gamma_i).$

Euclidean distance

The distance between points in $\mathbb{R}^2$ is defined using curves: for $P, Q \in \mathbb{R}^2$,

$$d_{euc}(P,Q) := \inf\left\{ \ell_{euc}(\gamma) \;\middle|\; \begin{array}{l} \gamma \text{ is a p.w.d. curve} \\ \text{from } P \text{ to } Q \end{array} \right\}.$$





Euclidean geodesics

From the curve definition we can recover the familiar formula for Euclidean distance.

<u>Prop</u>. The line segment $[P, Q]$ minimizes Euclidean length among $\underbrace{\text{p.w.d.}}_{\text{piecewise differentiable}}$ curves from P to Q.

(Proof.) HW. ◇

<u>Cor</u>. If $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, then

$$d_{euc}(P_0, P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

Metric spaces

Our first example of non-Euclidean geometry will involve a familiar set of points with an unfamiliar way of measuring distances.

A metric space is a pair (X, d), where X is a set and $d : X \times X \to [0, \infty)$ is a function, s.t.

① $d(P,P) = 0$, $\forall P \in X$;

② $d(P,Q) = 0$ iff $P = Q$;

③ $d(P,Q) = d(Q,P)$, $\forall P, Q \in X$;

④ $\underbrace{d(P,R) \leq d(P,Q) + d(Q,R)}_{\text{triangle inequality}}$, $\forall P, Q, R \in X$.

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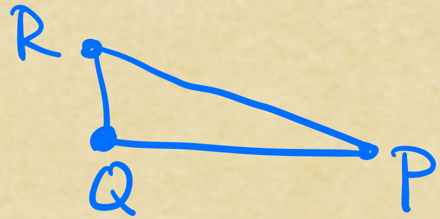


d is called the metric or distance function

If ①, ③, & ④ hold, we have a semi-metric.

Metric spaces

Metric spaces offer a good amount of structure because we can talk about limits & continuity.

- A sequence $(P_n) = P_1, P_2, \ldots, P_n, \ldots \in X$ is said to converge to a limit $L \in X$ if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $d(P_n, L) < \varepsilon$, for every $n \geq N$.

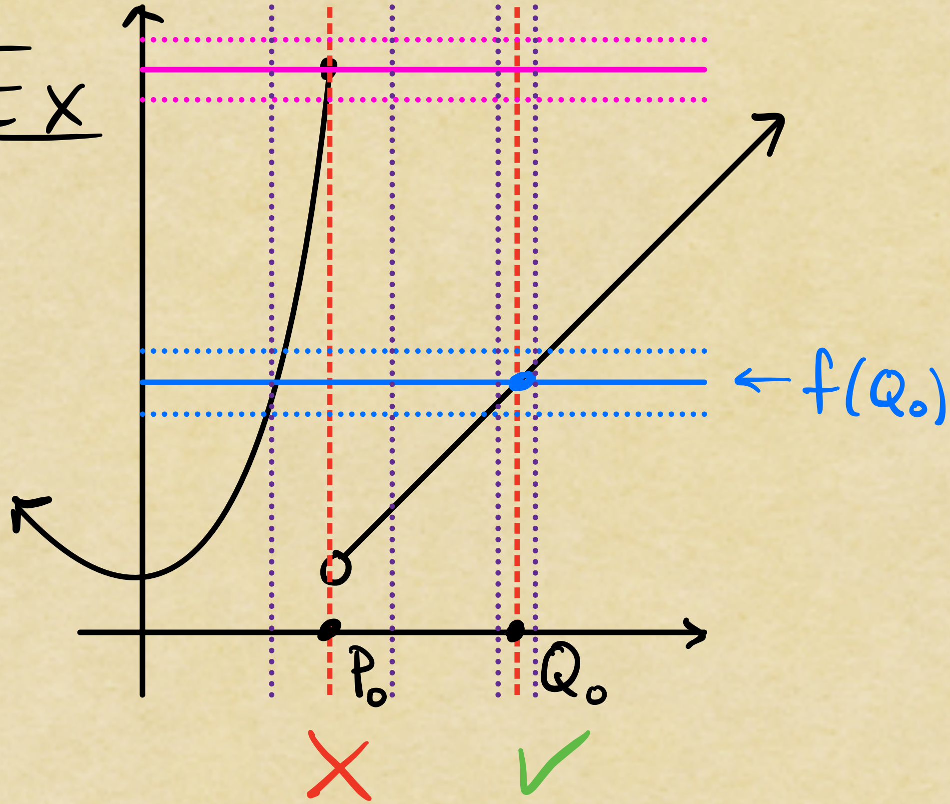
- A function $\varphi : X \to \tilde{X}$ btwn metric spaces (X, d) and $(\tilde{X}, \tilde{d})$ is called continuous at $P_0 \in X$ if, for every $\varepsilon > 0$, there is some $\delta > 0$ s.t.

$$d(P, P_0) < \delta \implies \tilde{d}(\varphi(P), \varphi(P_0)) < \varepsilon.$$

"Continuous" = continuous @ every $P_0 \in X$

Metric spaces

Ex



Vocab The ball with center $P_0 \in X$ and radius $r > 0$ is $B_d(P_0, r) = \{P \in X \mid d(P, P_0) < r\}$.

Isometries

A powerful way of studying the geometry of a metric space is to understand its group of symmetries. Here, a symmetry is an isometry.

An isometry btwn (X, d) and $(\tilde{X}, \tilde{d})$ is a bijection $\varphi : X \to \tilde{X}$ such that

$$\tilde{d}(\varphi(P), \varphi(Q)) = d(P, Q), \quad \forall\, P, Q \in X.$$

Check:
- the inverse φ^{-1} is an isometry
- φ is continuous
- compositions of isometries are isometries

(This is the group operation.)

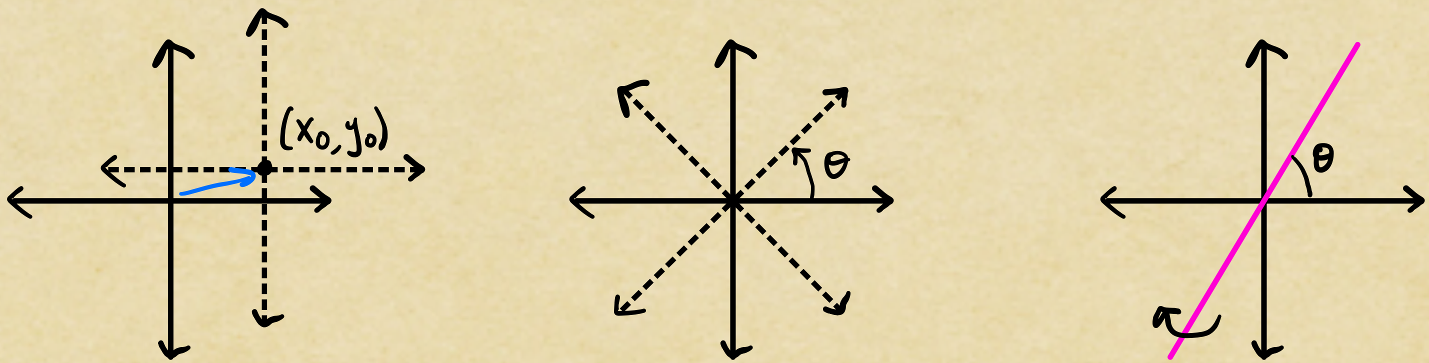
Isometries

Ex Here are some $\overbrace{\text{isometries of } (\mathbb{R}^2, d_{euc})}^{\text{i.e., } \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ preserving } d_{euc}}$:

translation : $\varphi(x,y) = (x + x_0, y + y_0)$

rotation : $\varphi(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

reflection : $\varphi(x,y) = (x\cos 2\theta + y\sin 2\theta, x\sin 2\theta - y\cos 2\theta)$



Fact: These generate all isometries of $(\mathbb{R}^2, d_{euc})$.

Isometries

Ex In complex coordinates:

translation: $\varphi(z) = z + z_0$

rotation: $\varphi(z) = e^{i\theta} z$

reflection: $\varphi(z) = e^{2i\theta} \bar{z}$

$$\boxed{\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy \\ e^{i\theta} &= \cos\theta + i\sin\theta \end{aligned}}$$

Prop. If φ is an isometry of $(\mathbb{R}^2, d_{euc}) = (\mathbb{C}, d_{euc})$, then there is a point $z_0 \in \mathbb{C}$ and an angle $\theta \in \mathbb{R}$ s.t. $\varphi(z) = e^{i\theta} z + z_0$ or $\varphi(z) = e^{2i\theta} \bar{z} + z_0$, for every $z \in \mathbb{C}$.

Next

We'll spend the rest of January on non-Euclidean geometries, starting with the hyperbolic plane.