Math 4803 January 31, 2024

LAST TIME

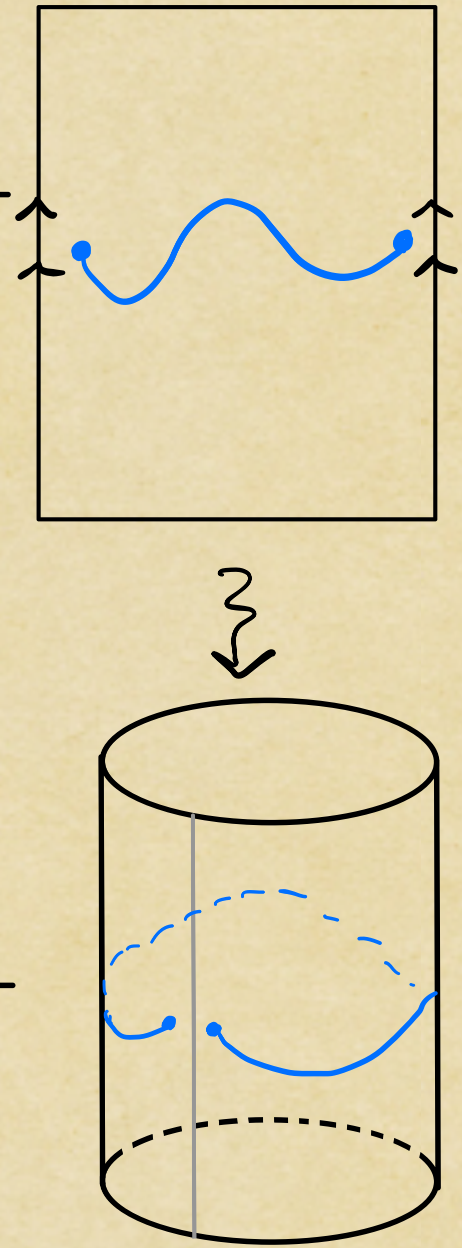
Spherical geometry provided our third fundamental 2D geometry, the others being Euclidean and hyperbolic.

TODAY

We'll build some surfaces by gluing pieces of the Euclidean plane.

Intuition for gluing constructions

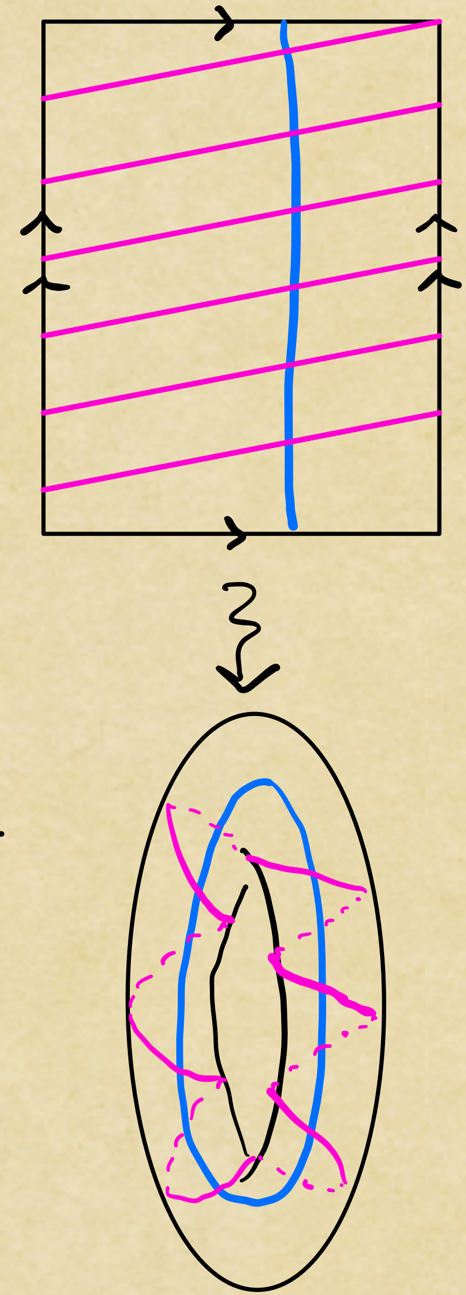
When we create a cylinder by rolling up a sheet of paper, distances btwn points may change, but arclengths do not.



Upshot: We're working with a new metric space on a global level, but still doing Euclidean geometry on a local level.

Intuition for gluing constructions

We can imagine using a similar construction to induce a metric on the torus.



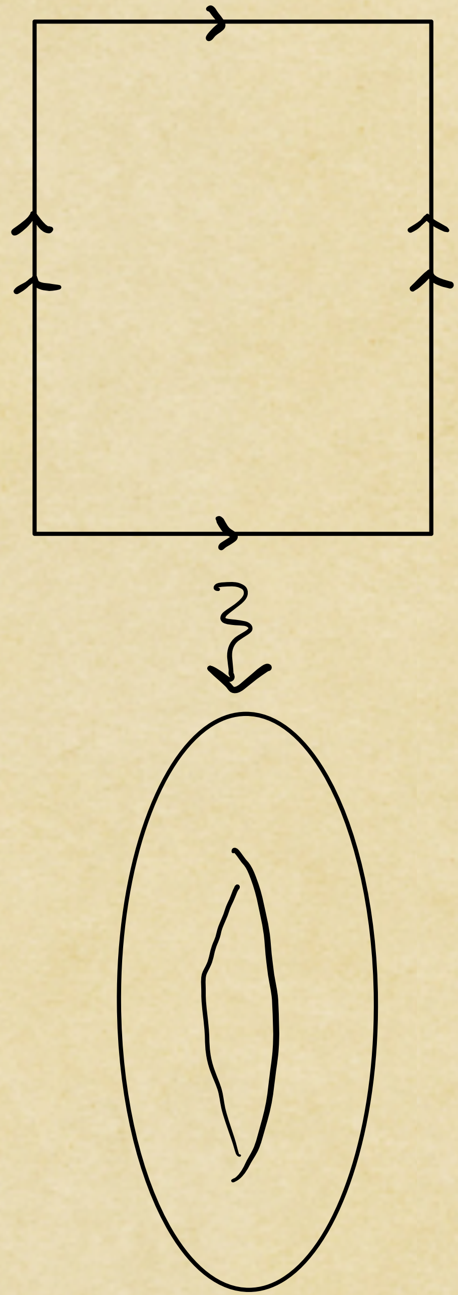
Visualizing in $\mathbb{R}^3$ doesn't work, since some arclengths would stretch, but we can simply declare that lengths (but not distances) are measured as in $(\mathbb{R}^2, d_{euc})$.

Intuition for gluing constructions

To make this rigorous we must determine:

(1) How to glue. That is, what is the set underlying our new metric space?

(2) How to measure distances. That is, what is our new metric?



Partitions

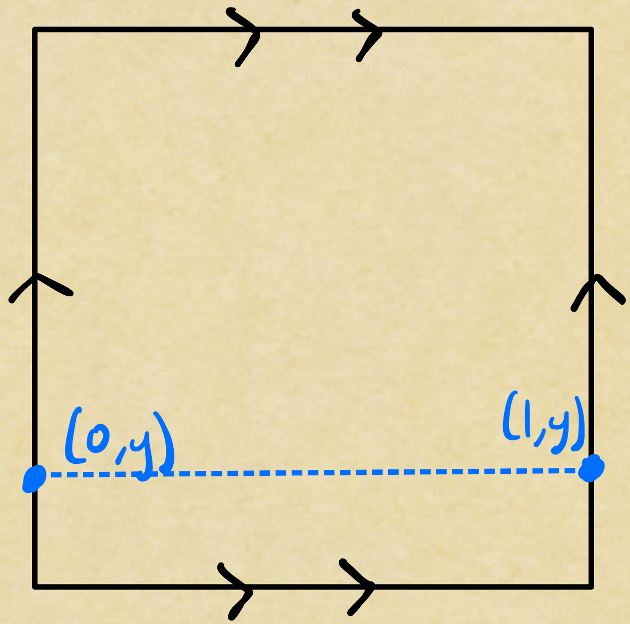
A partition of a set X is a collection $\overline{X}$ of subsets of X such that each element of X is contained in exactly one element of $\overline{X}$.

e.g., $\overline{X} = \{\{1,4\},\{2\},\{3,5\}\}$ is a partition of $X = \{1,2,3,4,5\}$

Given $P \in X$, we'll write $\overline{P}$ for the element of $\overline{X}$ which contains P. We'll write $P \sim Q$ for $P, Q \in X$ with $\overline{P} = \overline{Q}$.

Partitions

Ex



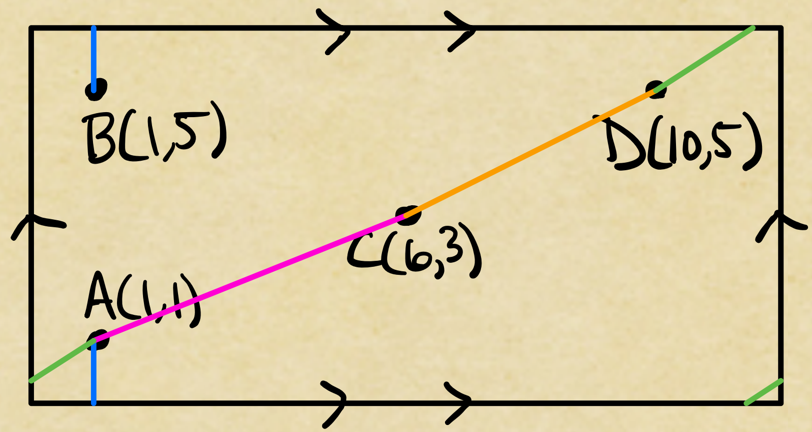
$X = [0,1] \times [0,1]$

The decorations on the edges determine a partition $\overline{X}$.

$$\rightsquigarrow \overline{(x,y)} = \begin{cases} \{(x,y)\}, & \text{if } 0 < x, y < 1 \\ \{(x,y), (1-x, y)\}, & \text{if } x = 0 \text{ or } x = 1 \text{ \& } 0 < y < 1 \\ \{(x,y), (x, 1-y)\}, & \text{if } y = 0 \text{ or } y = 1 \text{ \& } 0 < x < 1 \\ \{(0,0), (1,0), (0,1), (1,1)\}, & \text{if } (x,y) \text{ is a corner} \end{cases}.$$

Distances

Group work: Consider the partition of $[0,12] \times [0,6]$ depicted to the right. What *should* the following distances be? Why?



$$\bar{d}(A,B) = 2$$

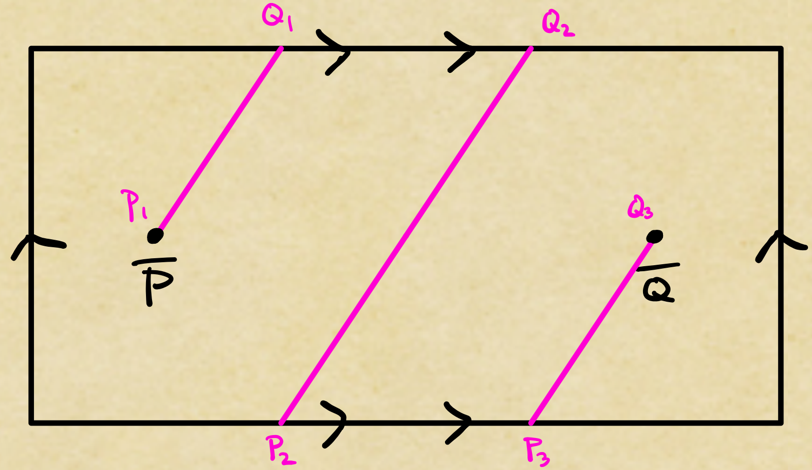
$$\bar{d}(A,C) = d(A,C) = \sqrt{29}$$

$$\bar{d}(C,D) = d(C,D) = \sqrt{20}$$

$$\bar{d}(A,D) = \sqrt{3^2 + 2^2} = \sqrt{13}$$

Discrete walks

A **discrete walk** w from $\overline{P} \in \overline{X}$ to $\overline{Q} \in \overline{X}$ is a finite sequence of points $P = P_1, Q_1, P_2, Q_2, \ldots, P_n, Q_n = Q$ such that $Q_i \sim P_{i+1}$, for $1 \leq i < n$.



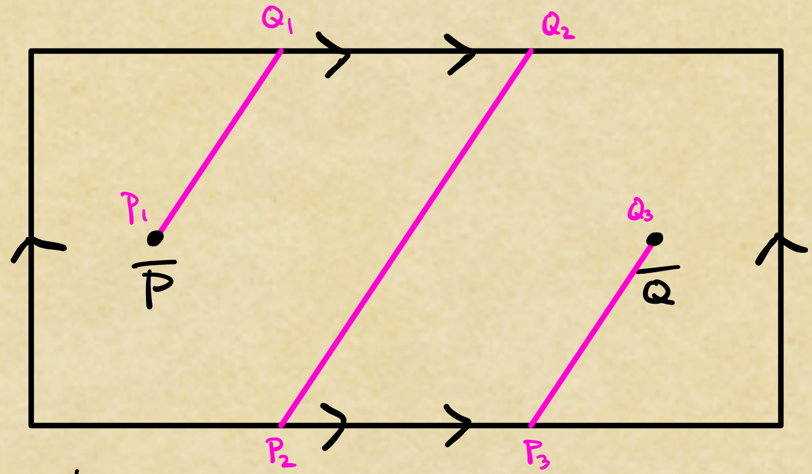
If (X, d) is a metric space, then the **length** of w is given by

$$\ell_d(w) := \sum_{i=1}^{n} d(P_i, Q_i).$$

Namely, discrete walks follow geodesics from P_is to Q_is and apparate/teleport from Q_i to P_{i+1}.

The quotient (semi-) metric

Discrete walks don't give all paths from $\overline{P}$ to $\overline{Q}$, but they give all candidates for the shortest paths.



Thus it makes sense to define $\overline{d} : \overline{X} \times \overline{X} \longrightarrow [0, \infty)$ by

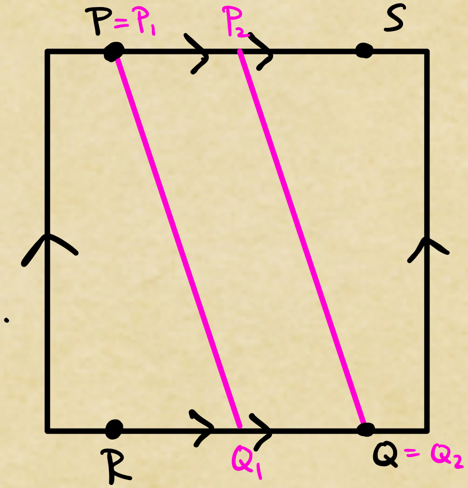
$$\overline{d}(\overline{P}, \overline{Q}) := \inf \left\{ \ell_d(w) \mid P \overset{w}{\rightsquigarrow} Q \text{ is a discrete walk} \right\}.$$

Lemma. The above function $\overline{d} : \overline{X} \times \overline{X} \to [0, \infty)$ is well-defined. Moreover, $\overline{d}$ is a semi-metric on $\overline{X}$.

The quotient (semi-) metric

Lemma. The above function $\overline{d}: \overline{X} \times \overline{X} \to [0,\infty)$ is well-defined. Moreover, $\overline{d}$ is a semi-metric on $\overline{X}$.

(Proof.) Showing that $\overline{d}$ is well-defined means verifying that $\overline{d}(\overline{P},\overline{Q})$ does not depend on the representatives P & Q of $\overline{P}$ & $\overline{Q}$.



So we have $P, Q, R, S \in X$ with $P \sim R$ and $Q \sim S$. We NTS $\overline{d}(\overline{P},\overline{Q}) = \overline{d}(\overline{R},\overline{S})$.

Let w be a d.w. $P = P_1, Q_1 \sim P_2, Q_2 \sim P_3, \dots, Q_n = Q$ from $\overline{P}$ to $\overline{Q}$. Then we get a d.w. w' from $\overline{R}$ to $\overline{S}$ by prepending $P_0 = R, Q_0 = R$ and appending $P_{n+1} = S, Q_{n+1} = S$.

Moreover, $\ell_d(w') = d(P_0, Q_0) + \ell_d(w) + d(P_{n+1}, Q_{n+1}) = \ell_d(w)$.

The quotient (semi-) metric

Likewise, every d.w. from $\overline{R}$ to $\overline{S}$ leads to a d.w. from $\overline{P}$ to $\overline{Q}$ of the same length. As a result, $\overline{d}(\overline{P},\overline{Q})=\overline{d}(\overline{R},\overline{S})$.

So $\overline{d}$ is a well-defined function.

To verify that $\overline{d}$ is a semi-metric, we must check:

(1) non-negativity: $\overline{d}(\overline{P},\overline{Q}) \geq 0$ & $\overline{d}(\overline{P},\overline{P})=0$, $\forall \overline{P},\overline{Q} \in \overline{X}$;

(2) symmetry: $\overline{d}(\overline{P},\overline{Q})=\overline{d}(\overline{Q},\overline{P})$, $\forall \overline{P},\overline{Q} \in \overline{X}$;

(3) triangle inequality: $\overline{d}(\overline{P},\overline{Q}) \leq \overline{d}(\overline{P},\overline{R})+\overline{d}(\overline{R},\overline{Q})$, $\forall \overline{P},\overline{Q},\overline{R} \in \overline{X}$.

Condition (1) follows quickly from our def'n of ℓ_d and the corresponding fact for d.

For condition (2) we notice that discrete walks can be reversed.

The quotient (semi-) metric

Finally, we obtain the triangle inequality via the usual concatenation argument. Namely, let w be a d.w.

$$P = P_1,\ Q_1 \sim P_2,\ Q_2 \sim P_3,\ \dots,\ Q_{n-1} \sim P_n,\ Q_n = Q$$

and let w' be a d.w. $Q = Q_1',\ R_1 \sim Q_2',\ R_2 \sim Q_3',\ \dots,\ R_{n-1} \sim Q_n',\ R_n = R$.

Then $\underbrace{P = P_1,\ Q_1 \sim P_2,\ \dots,\ Q_{n-1} \sim P_n,\ Q_n} \sim \underbrace{Q_1',\ R_1 \sim Q_2',\ \dots,\ R_{n-1} \sim Q_n',\ R_n = R}$ is a d.w. w'' from P to R, and $\ell_d(w'') = \ell_d(w) + \ell_d(w')$.

By passing to infima, we find that $\overline{d}(\overline{P}, \overline{R}) \leq \overline{d}(\overline{P}, \overline{Q}) + \overline{d}(\overline{Q}, \overline{R})$.

$\diamond$

There are examples (see exercise 4.1) where $\overline{d}(\overline{P}, \overline{Q}) = 0$ does not imply $\overline{P} = \overline{Q}$, and thus $\overline{d}$ is not always a metric.

The quotient (semi-)metric

If $\bar{d}$ does satisfy

$$\bar{d}(\bar{P},\bar{Q}) = 0 \implies \bar{P} = \bar{Q}, \quad \forall\, \bar{P}, \bar{Q} \in \bar{X},$$

then we say that we have a proper partition of (X,d), and call $\bar{d}$ the quotient metric on $\bar{X}$.

Regardless of this property, we call the map

$$\begin{aligned} \pi : X &\longrightarrow \bar{X} \\ P &\longmapsto \bar{P} \end{aligned}$$

the quotient map.

Exercise: For every $P, Q \in X$, $\bar{d}(\bar{P},\bar{Q}) \le d(P,Q)$, and thus π is continuous.