Math 4803 January 29, 2024

LAST TIME

Differentials and the hyperbolic norm.

TODAY

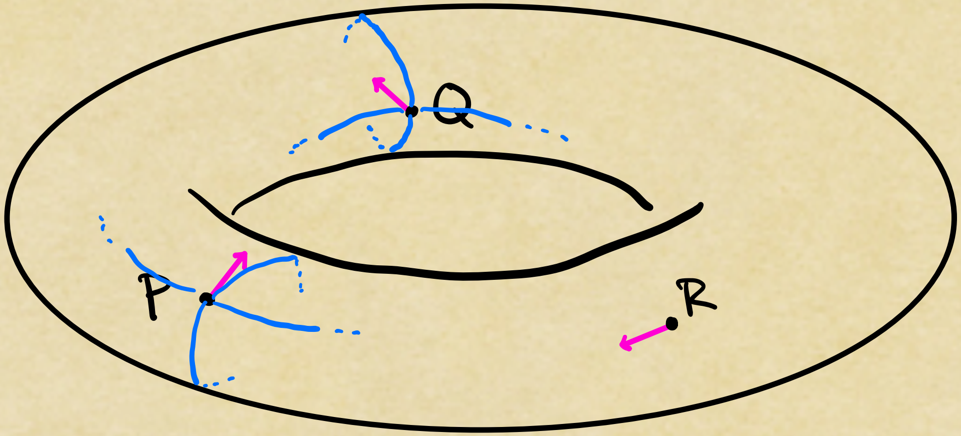
Spherical geometry.

Summary of $(\mathbb{H}^2, d_{hyp})$, $(\mathbb{R}^2, d_{euc})$, (S^2, d_{sph})

	$(\mathbb{H}^2, d_{hyp})$	$(\mathbb{R}^2, d_{euc})$	(S^2, d_{sph})
norm	$\lVert\vec{v}\rVert_{euc} / y$	$\sqrt{a^2+b^2}$	$\sqrt{a^2+b^2+c^2}$
length	—	$\ell_\bullet(\gamma) = \int_a^b \lVert\gamma'(t)\rVert_\bullet \, dt$	—
distance	—	$d_\bullet(P,Q) = \inf\{\ell_\bullet(\gamma) \mid P \overset{\gamma}{\rightsquigarrow} Q \text{ p.w.d.}\}$	—
geodesics	circular arcs centered on x-axis + vertical lines	straight lines	great circular arcs
isometries	(A)LFMs w/ real coeff.	$\vec{x} \mapsto A\vec{x} + \vec{v}$ s.t. $A^T A = I$	rotations & rotation-reflections

All three are isotropic.

Isotropy



The metric space $(\mathbb{R}^2, d_{euc})$ is isotropic b/c, $\forall$ $P, Q \in \mathbb{R}^2$ and $\forall$ $\vec{v}, \vec{w} \in \mathbb{R}^2$ with

$$\|\vec{v}\|_{euc} = \|\vec{w}\|_{euc},$$

$\exists$ isometry $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t.

$$\varphi(P) = Q \text{ \& } (D_P\varphi)(\vec{v}) = \vec{w}.$$

Same is true of $(\mathbb{H}^2, d_{hyp})$ & (S^2, d_{sph}).

What's the point?

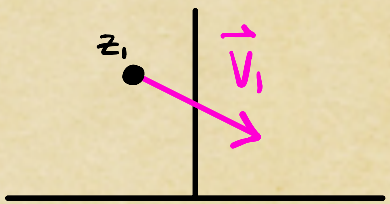
Isotropy is an exceptionally rare phenomenon — a geometry must be very symmetric to be isotropic.

The isotropy property for $(\mathbb{H}^2, d_{hyp})$

<u>Prop.</u> Given any points $z_1, z_2 \in \mathbb{H}^2$ and any vectors $\vec{v}_1, \vec{v}_2$ based at these points with $\|\vec{v}_1\|_{hyp} = \|\vec{v}_2\|_{hyp}$, there is an isometry φ of $(\mathbb{H}^2, d_{hyp})$ s.t.

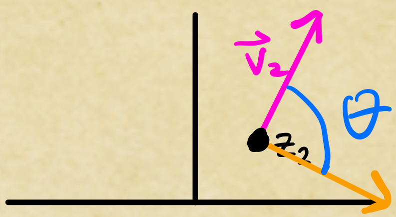
$$\varphi(z_1) = z_2 \text{ \& } (D_{z_1}\varphi)(\vec{v}_1) = \vec{v}_2 .$$

(Proof.) First, let θ be the angle btwn $\vec{v}_1$ & $\vec{v}_2$ as



we'd measure it in <u>Euclidean geom</u>.

Now find $c, d \in \mathbb{R}$ s.t.



$$\left.\begin{array}{l} c x_1 + d = \cos\left(\frac{\theta}{2}\right) \\ \text{\& } -c y_1 = \sin\left(\frac{\theta}{2}\right) \end{array}\right\} \rightarrow \text{i.e., } c z_1 + d = e^{-i\frac{\theta}{2}},$$

where $z_1 = x_1 + i y_1$.

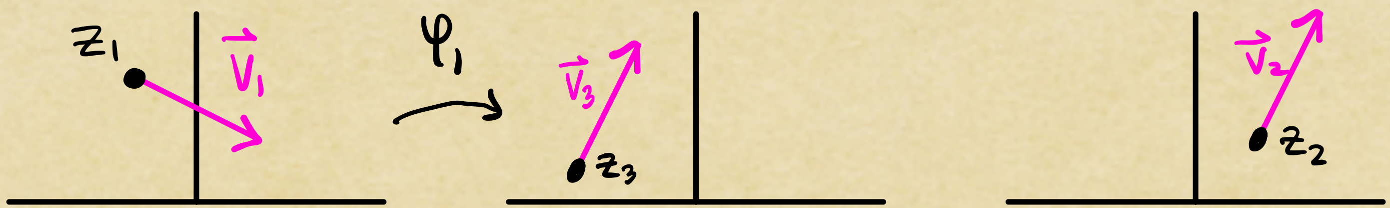
The isotropy property for $(\mathbb{H}^2, d_{hyp})$

Once we have $c, d \in \mathbb{R}$ s.t. $cz_1 + d = e^{-i\frac{\theta}{2}}$, pick $a, b \in \mathbb{R}$ s.t. $ad - bc = 1$.

Now let $\varphi_1(z) = \dfrac{az+b}{cz+d}$ Then

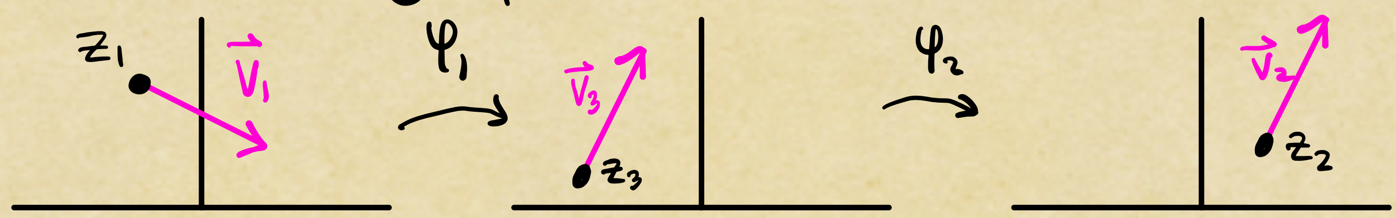
$$D_{z_1}\varphi(v) = \frac{1}{(cz_1+d)^2} v = \frac{1}{(e^{-i\theta/2})^2} v = e^{i\theta} v$$

is rotation by an angle of θ.



So $\vec{v}_3 := D_{z_1}\varphi_1(\vec{v}_1)$ is positively parallel $\vec{v}_2$ in the Euclidean sense. We let $z_3 := \varphi_1(z_1)$.

The isotropy property for $(\mathbb{H}^2, d_{hyp})$



Next, we choose $\varphi_2 : \mathbb{H}^2 \to \mathbb{H}^2$ to be a composition of a homothety and a translation s.t. $\varphi_2(z_3) = z_2$.

Check: This ensures that $D_{z_3}\varphi_2$ is a homothety.

Finally, let $\varphi = \varphi_2 \circ \varphi_1$. Then $\varphi(z_1) = \varphi_2(z_3) = z_2$.

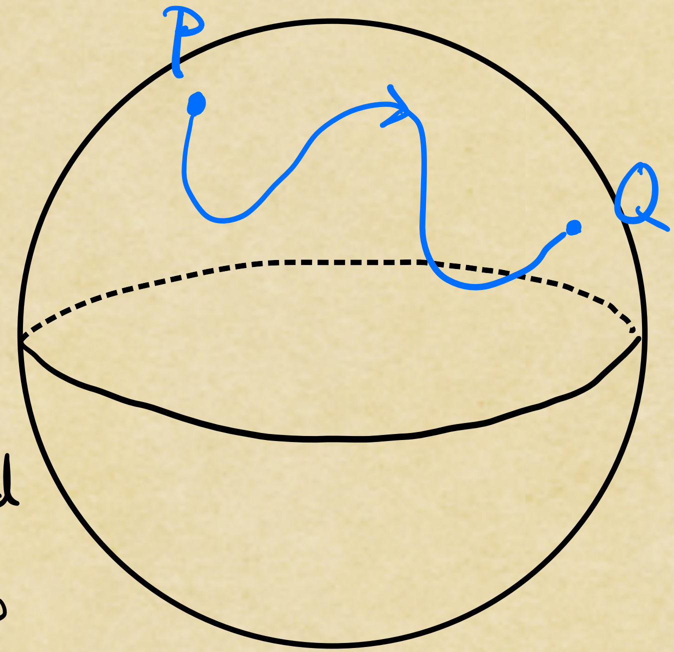
Moreover, $D_{z_1}\varphi = D_{z_3}\varphi_2 \circ D_{z_1}\varphi_1$ preserves the hyperbolic norm, so $\|D_{z_1}\varphi(\vec{v}_1)\|_{hyp} = \|\vec{v}_1\|_{hyp} = \|\vec{v}_2\|_{hyp}$.

Since $D_{z_1}\varphi(\vec{v}_1)$ is positively parallel to $\vec{v}_2$,

$$D_{z_1}\varphi(\vec{v}_1) = \vec{v}_2.$$

$\diamond$

The metric space (S^2, d_{sph})



As a set, the 2-dimensional sphere is

$$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3.$$

For any p.w.d. curve γ parametrized by $t \mapsto (x(t), y(t), z(t))$, $a \le t \le b$ we have

$$\ell_{euc}(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

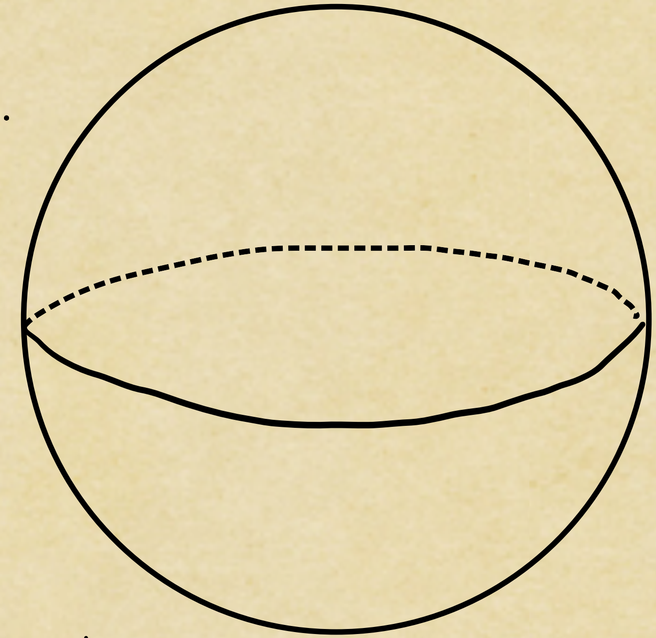
We define the spherical distance d_{sph} by

$$d_{sph}(P,Q) := \inf\{\ell_{euc}(\gamma) \mid P \overset{\gamma}{\rightsquigarrow} Q \text{ in } S^2\}.$$

<u>Note</u>: $d_{sph}(P,Q) \ge d_{euc}(P,Q)$, $\forall P, Q \in S^2$

The metric space (S^2, d_{sph})

Prop $d_{sph} : S^2 \times S^2 \longrightarrow S^2$ is a metric.



(Proof idea). For all $P, Q, R \in S^2$, we need:

① $d_{sph}(P,P) = 0$;

② $d_{sph}(P,Q) = 0 \Rightarrow P = Q$;

③ $d_{sph}(Q,P) = d_{sph}(P,Q)$;

④ $d_{sph}(P,Q) \leq d_{sph}(P,R) + d_{sph}(R,Q)$.

For ①, let γ be constant. For ③, notice that we can reverse the direction any p.w.d. curve $P \rightsquigarrow Q$.

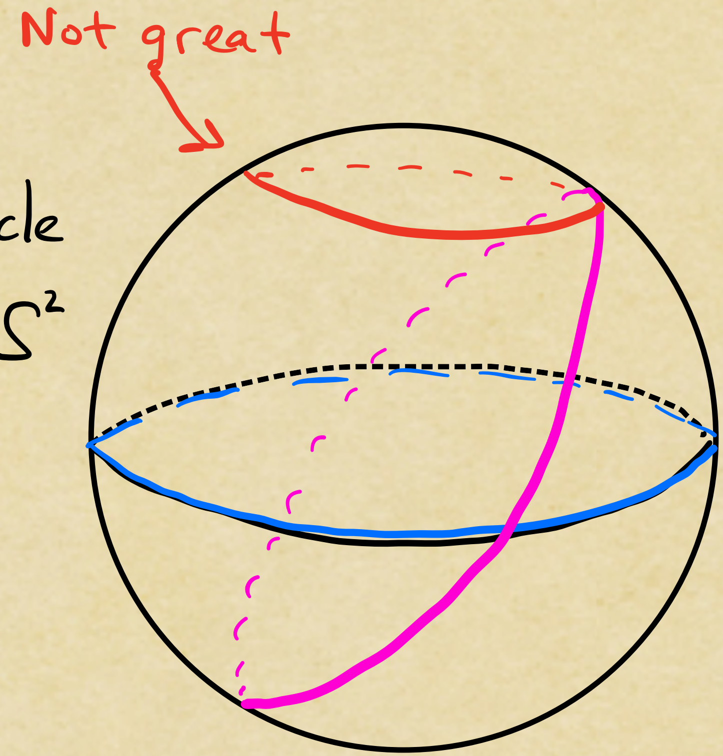
For ④, concatenate curves, as we did in hyperbolic geometry.

For ②, notice that $d_{sph}(P,Q) = 0 \Rightarrow d_{euc}(P,Q) = 0 \Rightarrow P = Q$. $\diamond$

Geodesics

A great circle in S^2 is a circle which lies in the intersection of S^2 and a plane thru the origin.

A great circle arc is an arc contained in a great circle.

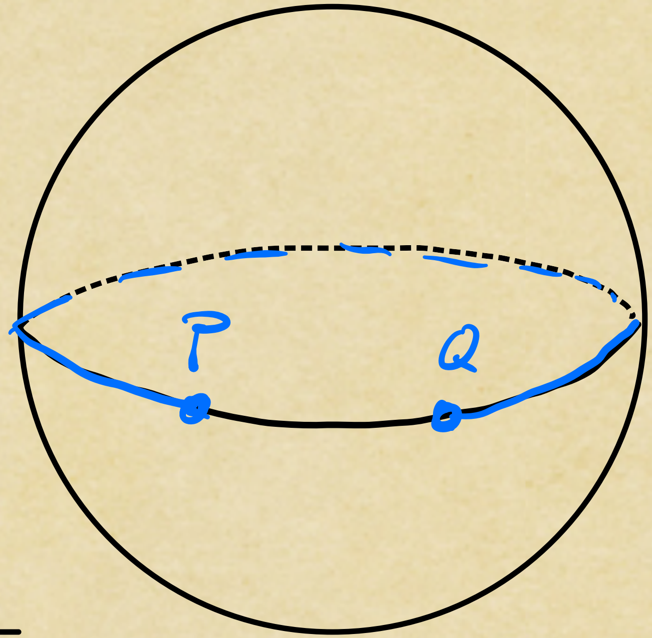


Thm. The geodesics of (S^2, d_{sph}) are the great circle arcs. The shortest curve(s) from P to Q in S^2 are the great circle arc(s) of length $\leq \pi$.

(Proof idea.) Using spherical coords., the strategy is as in $\mathbb{H}^2$: investigate an integral in a simple case, then apply an isometry to other cases. See exercise 3.1. $\diamond$

Geodesics

Some observations:

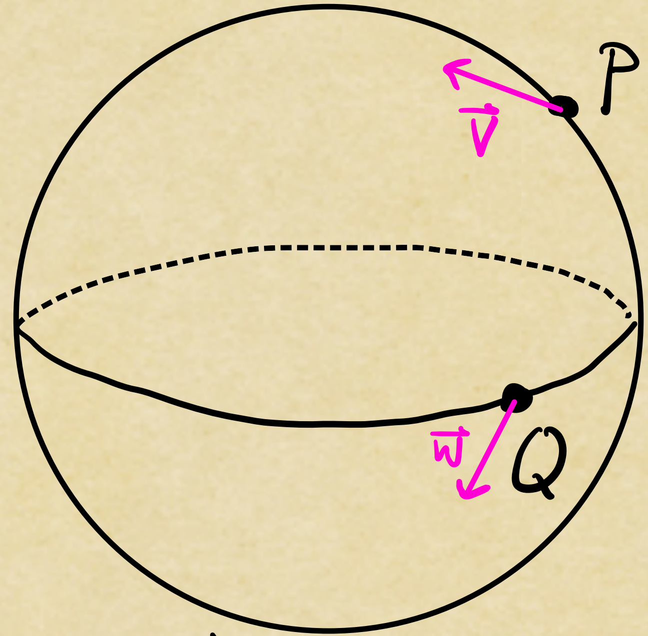


- (S^2, d_{sph}) has a finite diameter.
  Namely, $d_{sph}(P,Q) \leq \pi$, $\forall P,Q \in S^2$.
- Unlike in $(\mathbb{R}^2, d_{euc})$ & $(\mathbb{H}^2, d_{hyp})$,
  geodesic $\not\Rightarrow$ shortest curve

  (But the implication holds for sufficiently close points along a geodesic.)
- Also unlike the Euclidean & hyperbolic cases, spherical geometry admits closed geodesics, i.e., geodesics with parametrizations which are periodic.

Isotropy

Any rotation $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ about a line L thru the origin preserves arc lengths of curves in $\mathbb{R}^3$.



Moreover, $\varphi(S^2) = \underline{S^2}$, so we have an isometry of (S^2, d_{sph}).

(arclengths preserved $\Rightarrow$ distances preserved)

Thm. Pick $P, Q \in S^2$ and unit vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ which are tangent* to S^2 at P, Q, respectively. There is an isometry $\varphi : S^2 \to S^2$ such that

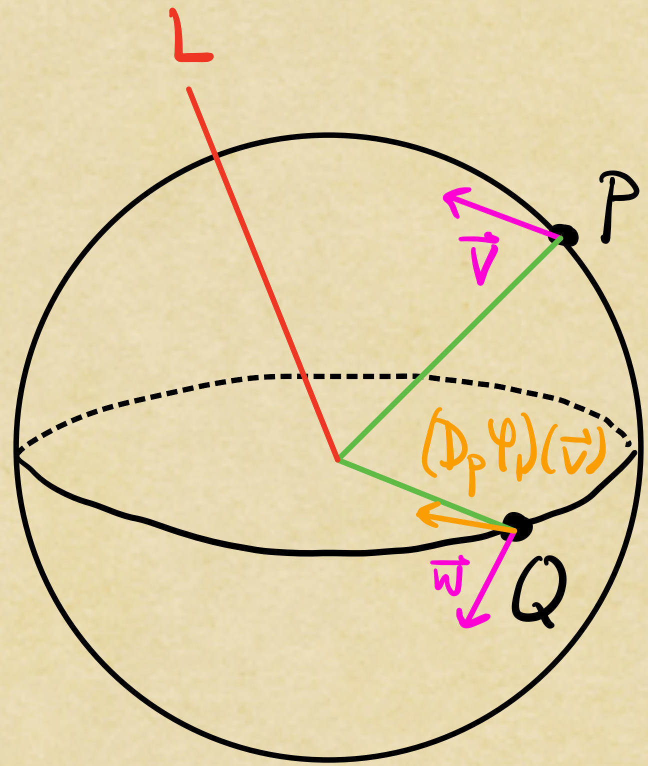
$$\varphi(P) = Q \quad \text{and} \quad (D_P \varphi)(\vec{v}) = \vec{w} .$$

*We haven't declared what this means, but also won't worry much about it.

Isotropy

Thm. Pick $P, Q \in S^2$ and unit vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ which are tangent to S^2 at P, Q, respectively. There is an isometry $\varphi : S^2 \to S^2$ such that

$$\varphi(P) = Q \quad \text{and} \quad D_P\varphi(\vec{v}) = \vec{w}.$$



(Proof). First we let φ_1 be a rotation about the line thru the origin which is perpendicular to both OP and OQ. Specifically, we ensure that $\varphi_1(P) = Q$.

Next, φ_2 is the rotation about OQ which satisfies $(D_Q\varphi_2 \circ D_P\varphi_1)(\vec{v}) = \vec{w}$. (Note that $\varphi_2(Q) = Q$.)

Check: $\varphi_2 \circ \varphi_1$ is what we want. $\diamond$

All isometries of (S^2, d_{sph})

We've decided that a rotation around a line L thru the origin gives an isometry of (S^2, d_{sph}).

Another isometry is given by reflecting across the plane thru the origin $\perp$ L.

(Proof sketched in exercises, plus a different explanation @ end of these slides.)

The composition of a rotation about L and a reflection across a plane thru O which is $\perp$ the same L is called a rotation-reflection.

Thm. Every isometry of (S^2, d_{sph}) is either a rotation or a rotation-reflection about some line L thru the origin.

Where are we headed?

We now have the three fundamental 2D geometries we'll need. From these building blocks we'll construct interesting metric spaces with the same local structure but very different global behavior.

Appendix*: Rotations & reflections are isometries

Here's a way of thinking about rotations and reflections which is slightly different from the book's proof.

First, a map $\varphi : S^2 \to S^2$ will preserve distances iff the differential $D_P\varphi$ preserves Euclidean norms, for every $P \in S^2$.

This is because

$$d_{sph}(\varphi(P), \varphi(Q)) = \inf\left\{ \int_a^b \|(\varphi\circ\gamma)'(t)\|_{euc}\, dt \;\middle|\; P \overset{\gamma}{\rightsquigarrow} Q \right\}$$

$$= \inf\left\{ \int_a^b \|D_{\gamma(t)}\varphi\,(\gamma'(t))\|_{euc}\, dt \;\middle|\; P \overset{\gamma}{\rightsquigarrow} Q \right\}$$

$$= \inf\left\{ \int_a^b \|\gamma'(t)\|_{euc}\, dt \;\middle|\; P \overset{\gamma}{\rightsquigarrow} Q \right\} = d_{sph}(P, Q).$$

* Totally optional. Just adding this in case you care.

Appendix: Rotations & reflections are isometries

Next, if $\overline{\varphi}: \mathbb{R}^3 \to \mathbb{R}^3$ is linear and satisfies $\overline{\varphi}(S^2) = S^2$, then the restriction $\varphi = \overline{\varphi}|_{S^2}$ has $D_p\varphi = \varphi$, for all $p \in S^2$.

(Justified using curves approach to differentials.)

Moreover, if the matrix representation A for $\overline{\varphi}$ in an orthonormal basis satisfies $A^TA = I$, then $\overline{\varphi}$ preserves Euclidean norms:

Sliding a matrix across a dot product requires a transpose

$$
\begin{aligned}
\|\overline{\varphi}(\vec{v})\|_{euc}^2 &= \|A\vec{v}\|_{euc}^2 \\
&= (A\vec{v})\cdot(A\vec{v}) = \vec{v}\cdot(A^TA\vec{v}) \\
&= \vec{v}\cdot\vec{v} = \|\vec{v}\|_{euc}^2.
\end{aligned}
$$

So a linear map $\overline{\varphi}: \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix representation A satisfies $A^TA = I$ induces an isometry $\varphi: S^2 \to S^2$.

Appendix: Rotations & reflections are isometries

Finally, let L be a line thru the origin and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be an orthonormal basis for $\mathbb{R}^3$ with $\vec{v}_1 \parallel L$.

Rotation R_θ about L thru an angle θ and reflection r_L across the plane orthogonal to L are linear maps with matrix representations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in this basis. These satisfy $A^T A = I$ and thus the same is true of the matrix representations of R_θ and r_L in the standard basis.

So R_θ & r_L are isometries of (S^2, d_{sph}).