Math 4803 January 24, 2024

LAST TIME

We showed that all isometries of $(\mathbb{H}^2, d_{hyp})$ are of the form $\varphi(z) = \dfrac{az+b}{cz+d}$ or $\varphi(z) = \dfrac{c\bar{z}+d}{a\bar{z}+b}$, with a,b,c,d real and satisfying $ad-bc=1$. Allowing a,b,c,d to be complex gives us (anti)linear fractional map $\varphi : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$.

TODAY

We'll study the differentials of these maps, which allow us to understand their effect on tangent vectors.

(Anti)linear fractional maps

Recall that a linear fractional map has the form

$$\varphi(z) = \frac{az+b}{cz+d}, \quad (a,b,c,d \in \mathbb{C} \,;\, ad-bc \neq 0)$$

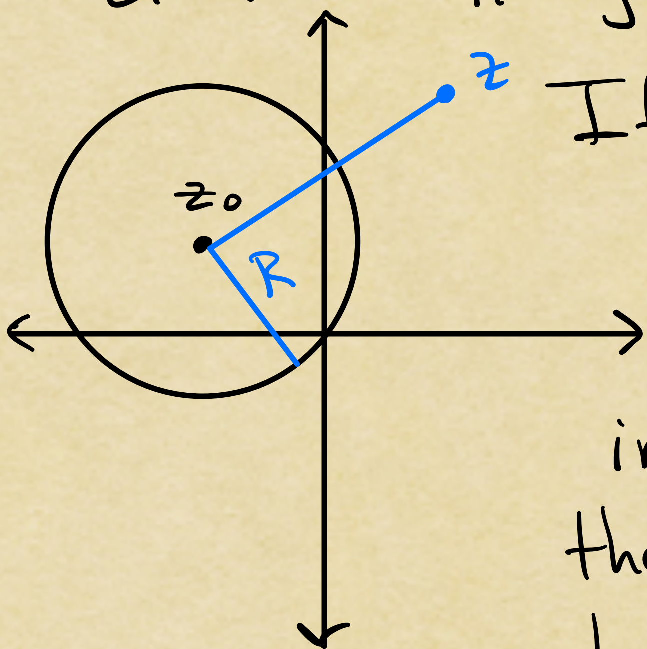
while an antilinear fractional map has the form

$$\varphi(z) = \frac{c\bar{z}+d}{a\bar{z}+b}, \quad (a,b,c,d \in \mathbb{C} \,;\, ad-bc \neq 0).$$

<u>Lemma</u>. Every (A)LFM is a composition of homotheties, translations, rotations, and inversions across the unit circle.

Other inversions

In $\hat{\mathbb{C}}$, inversions make sense across any circle — not just the unit circle.



If the center is z_0 & radius R, then

$$z \mapsto z_0 + \frac{R^2}{|z-z_0|^2}(z-z_0)$$

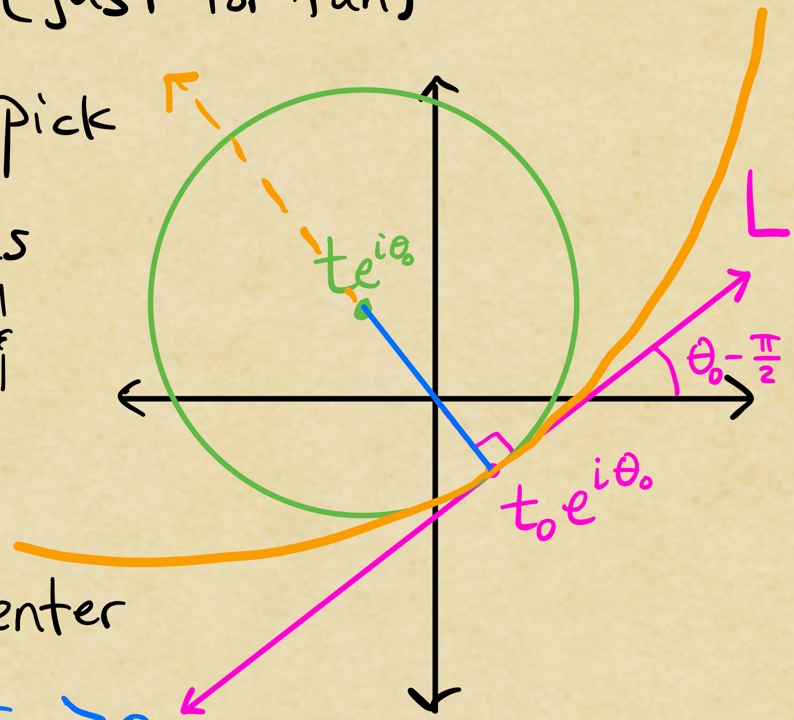
inverts $\hat{\mathbb{C}}$ across the circle, taking z to the point on the same ray out of z_0 as z, but at a Euclidean distance $R^2/|z-z_0|^2$ ________.

<u>Exercise</u>. Inversion is an ALFM. Namely,

$$\varphi(z) = \frac{z_0\bar{z} + (R^2 - |z_0|^2)}{\bar{z} - \bar{z}_0}.$$

Reflections as inversions (just for fun)

For any line $L \subset \mathbb{C}$ we can pick $t_0 < 0$ and $\theta_0 \in \mathbb{R}$ s.t. L makes an angle $\theta_0 - \frac{\pi}{2}$ w/ the x-axis & passes thru $t_0 e^{i\theta_0}$.



For any $t > 0$, the circle w/ center $z_0 = te^{i\theta_0}$ and radius $R = t - t_0 > 0$ passes thru $t_0 e^{i\theta_0}$ and is tangent to L.

Inversion over this circle is

$$\frac{z_0 \bar{z} + (R^2 - |z_0|^2)}{\bar{z} - \bar{z}_0} = \frac{te^{i\theta_0}\bar{z} + ((t-t_0)^2 - t^2)}{\bar{z} - te^{-i\theta_0}}$$

As $t \to \infty$ this tends to $-e^{2i\theta_0}\bar{z} + 2t_0 e^{i\theta_0}$ = refl. across L.

Differentials

Our next goal is to understand how vectors transform under (A)LFMs $\varphi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

In multivariable calculus you learned about the Jacobian determinant of a transformation

$$F: U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

Namely, if $F(x,y) = (f(x,y), g(x,y))$, then

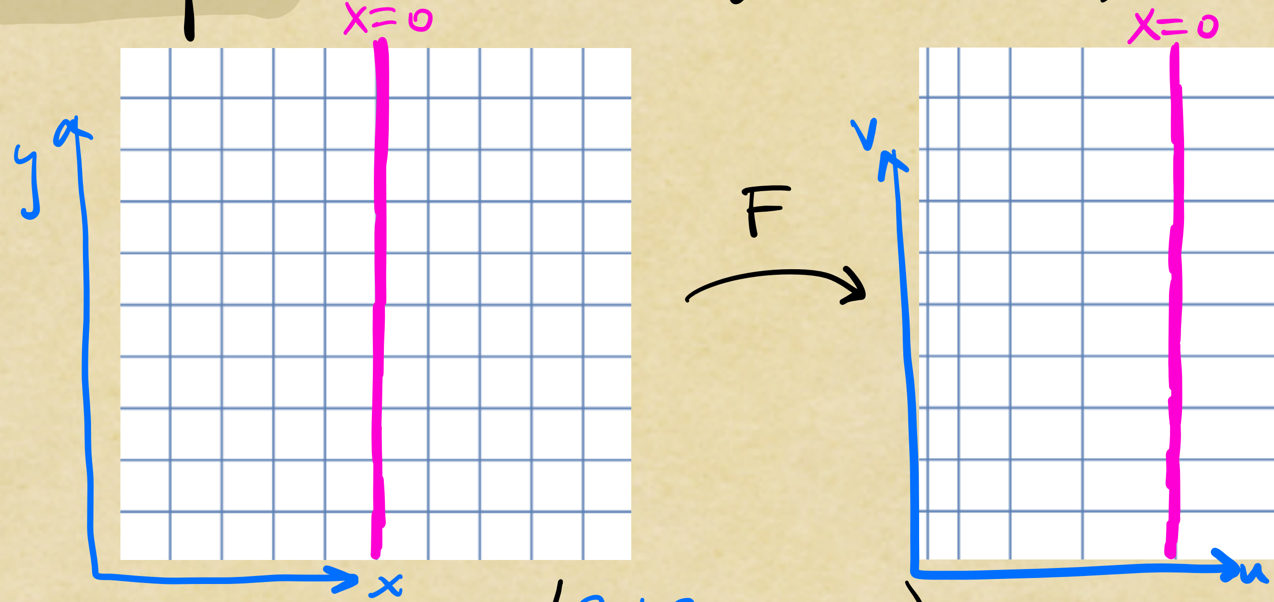
$$|J_F| = \left| \det \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \right|$$

gives a scale factor due to F.

We won't require $|J_F| \neq 0$.

Differentials

Example: Consider $F(x,y) = (2x + x^2, y)$, $-1 < x, y < 1$.

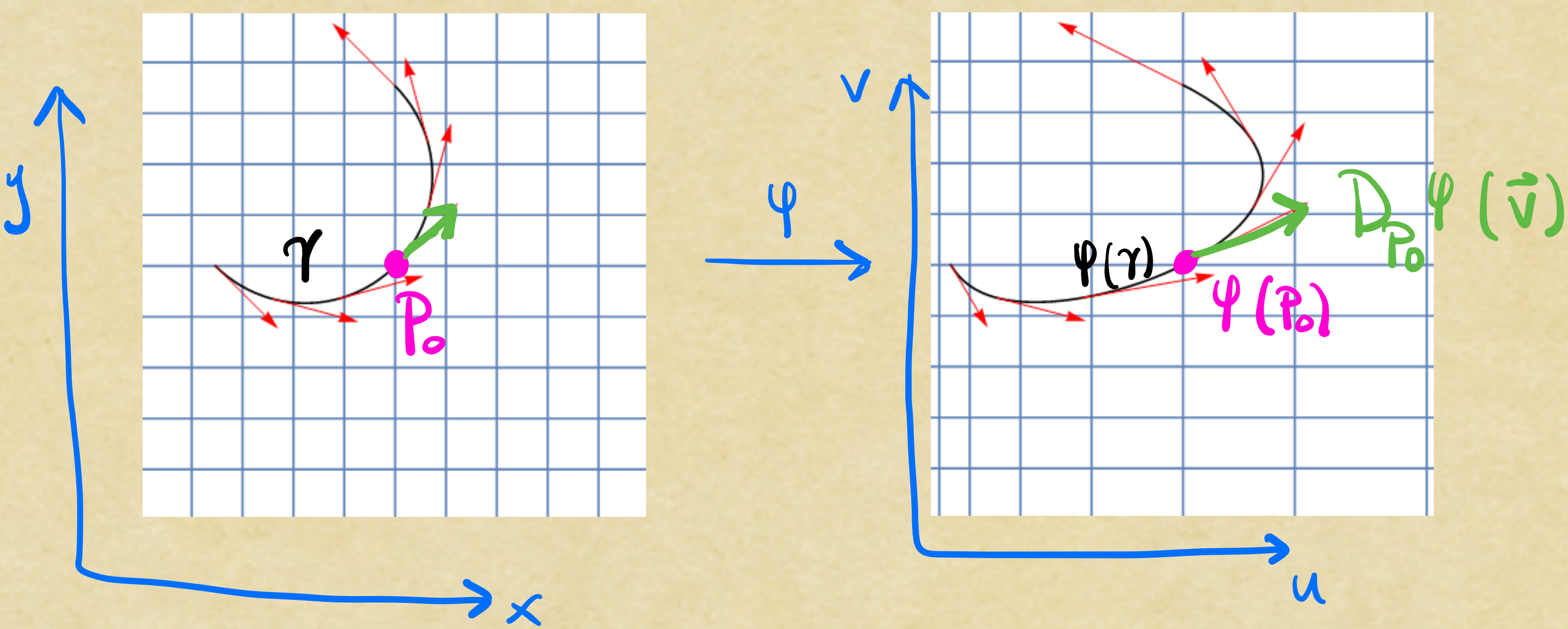


We have $J_F = \begin{pmatrix} 2+2x & 0 \\ 0 & 1 \end{pmatrix}$, so $|J_F| = |2+2x| = 2+2x$

Notation: For $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, we'll write $D_{P_0}\varphi$ for the <u>differential</u> of φ at P_0, the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{pmatrix} f_x(P_0) & f_y(P_0) \\ g_x(P_0) & g_y(P_0) \end{pmatrix}$.

Differentials

Lemma. If $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is diff'able, $P_0 \in \operatorname{int}(U)$, and γ is a parametrized curve in U thru P_0 which has tangent vector $\vec{v}$ at P_0, then $\varphi(\gamma)$ is a parametrized curve in $\varphi(U)$ thru $\underline{\varphi(P_0)}$ with tangent vector $\underline{D_{P_0}\varphi(\vec{v})}$ there.



Differentials

Corollary. Differentials satisfy a chain rule.
Namely, if $\varphi : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
& $\psi : V \subset \mathbb{R}^2 \to \mathbb{R}^2$ $(V \subset \operatorname{int}(\varphi(U)))$
are diff'able, then

$$D_{p_0}(\psi \circ \varphi) = D_{\varphi(p_0)}\psi \circ D_{p_0}\varphi .$$

Remark. A good way to unwind this is via matrix multiplication:

$$J_{\psi \circ \varphi} = J_\psi \cdot J_\varphi$$

Differentials of (A)LFMs

Corollary. The LFM $\varphi(z) = \dfrac{az+b}{cz+d}$ has differential $D_{z_0}\varphi : \mathbb{C} \to \mathbb{C}$ $(z_0 \neq -d/c)$ given by

$$D_{z_0}\varphi(v) = \frac{ad-bc}{(cz_0+d)^2}\, v \, ,$$

⚠ This is a cx number!

for every $v \in \mathbb{C}$. Similarly, the ALFM $\psi(z) = \dfrac{c\bar{z}+d}{a\bar{z}+b}$ has differential $D_{z_0}\psi : \mathbb{C} \to \mathbb{C}$ $(z_0 \neq -\bar{b}/\bar{a})$ given by

$$D_{z_0}\psi(v) = \frac{ad-bc}{(a\bar{z}_0+b)^2}\, \bar{v} \, ,$$

for every $v \in \mathbb{C}$.

(Proof.) See book. Idea is to consider $t \mapsto z_0 + tv$ at $t=0$. ◇

Differentials of (A)LFMs

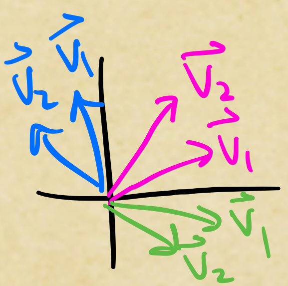
Note that $v \longmapsto \frac{ad-bc}{(cz_0+d)^2} v$ is the composition of a homothety and a rotation. (We need both b/c the coefficients might not be real.) Similarly, $v \longmapsto \frac{ad-bc}{(a\bar{z}_0+b)^2}\bar{v}$ consists of a homothety and a reflection.

Upshot: LFMs preserve angles & orientations

while ALFMs preserve angles & reverse orientations

i.e., $\measuredangle(D_{z_0}\varphi(\vec{v}_1), D_{z_0}\varphi(\vec{v}_2)) = \measuredangle(\vec{v}_1, \vec{v}_2)$

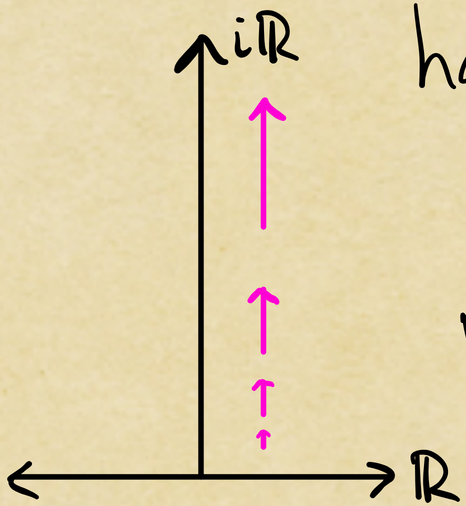
$\measuredangle(D_{z_0}\psi(\vec{v}_1), D_{z_0}\psi(\vec{v}_2)) = \measuredangle(\vec{v}_2, \vec{v}_1)$



The hyperbolic norm

In hyperbolic geometry, angles btwn vectors are measured just as in Euclidean geometry. However, magnitudes are computed differently, and the computation depends on the location of the vector: a vector $\vec{v}$ based at $z_0 \in \mathbb{H}^2$ has hyperbolic norm

$$\|\vec{v}\|_{hyp} = \frac{1}{\mathrm{Im}(z_0)} \|\vec{v}\|_{euc}$$



Why? Because* this makes the formula

$$\ell_{hyp}(\gamma) = \int_a^b \|\gamma'(t)\|_{hyp}\, dt$$

work.

*Of course it's slightly deeper than just pattern matching. For a more careful explanation, ask on Discord.

The hyperbolic norm

We already know that isometries of $(\mathbb{H}^2, d_{hyp})$ preserve angles (up to sign). They also preserve the hyperbolic norm.

<u>Lemma</u>. If $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry of $(\mathbb{H}^2, d_{hyp})$ and $\vec{v} \in \mathbb{C}$ is a vector based at $z_0 \in \mathbb{H}^2$, then

$$\| D_{z_0}\varphi(\vec{v}) \|_{hyp} = \|\vec{v}\|_{hyp}.$$

(Proof sketch.) Assume $\varphi(z) = \dfrac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$, $ad-bc=1$.

Then $D_{z_0}\varphi(\vec{v}) = \dfrac{1}{(cz_0+d)^2}\vec{v} \Rightarrow \|D_{z_0}\varphi(\vec{v})\|_{euc} = \dfrac{1}{|cz_0+d|^2}\|\vec{v}\|_{euc}$

Now check that $\mathrm{Im}(\varphi(z_0)) = \dfrac{1}{|cz_0+d|^2} \cdot \mathrm{Im}(z_0)$.

Then $\|D_{z_0}\varphi(\vec{v})\|_{hyp} = \dfrac{1}{\mathrm{Im}(\varphi(z_0))}\|D_{z_0}\varphi(\vec{v})\|_{euc} = \dfrac{1}{\mathrm{Im}(z_0)}\|\vec{v}\|_{euc}$

$= \|\vec{v}\|_{hyp}$ ✓

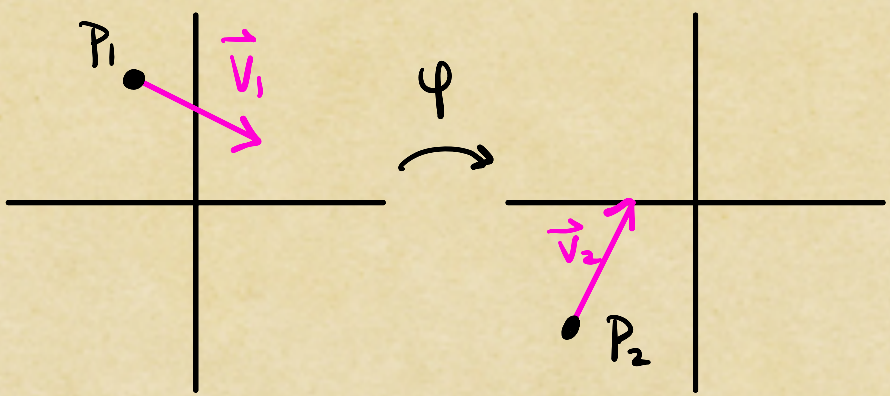
The antilinear case is similar. $\diamond$

The isotropy property

Finally, we'll show that $(\mathbb{H}^2, d_{hyp})$ is just as symmetric as $(\mathbb{R}^2, d_{euc})$.

Fact: Given any points $P_1, P_2 \in \mathbb{R}^2$ and any vectors $\vec{v}_1, \vec{v}_2$ based at these points with $\|\vec{v}_1\|_{euc} = \|\vec{v}_2\|_{euc}$, there is an isometry φ of $(\mathbb{R}^2, d_{euc})$ s.t.

$$\varphi(P_1) = P_2 \ \text{ \& } \ D_{P_1}\varphi(\vec{v}_1) = \vec{v}_2.$$



This is called the **isotropy property** of Euclidean geometry.