Math 4803 January 22, 2024

LAST TIME

In $(\mathbb{H}^2, d_{hyp})$, the shortest curve connecting P to Q is the circular arc with endpoints P & Q with center on the x-axis, where we understand vertical lines as such circles.

TODAY

We'll characterize all isometries of $(\mathbb{H}^2, d_{hyp})$ and take a small detour to consider these as self-maps of the Riemann sphere.

More isometries of $(\mathbb{H}^2, d_{hyp})$

For the rest of Ch. 2, $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$.

Recall the isometries we've already defined:

horizontal translation: $z \longmapsto z + x_0$ must be real

homothety: $z \longmapsto \lambda z, \quad \lambda > 0$

standard inversion: $z \mapsto \frac{1}{\bar{z}}$

Lemma. Pick $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

The maps

$$z \mapsto \frac{az + b}{cz + d} \quad \text{and} \quad z \mapsto \frac{c\bar{z} + d}{a\bar{z} + b}$$

are isometries of $(\mathbb{H}^2, d_{hyp})$.

More isometries of $(\mathbb{H}^2, d_{hyp})$

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are isometries of $(\mathbb{H}^2, d_{hyp})$.

(Proof.) Suppose $a \neq 0$ and consider

$$\underset{\text{hor. trans.}}{\varphi_1(z) = z + \frac{b}{a}}, \quad \underset{\text{std. inversion}}{\varphi_2(z) = \frac{1}{\bar{z}}}, \quad \underset{\text{homothety}}{\varphi_3(z) = \frac{z}{a^2}}, \quad \underset{\text{hor. trans.}}{\varphi_4(z) = z + \frac{c}{a}}.$$

Then $(\varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1)(z) = (\varphi_4 \circ \varphi_3 \circ \varphi_2)\left(z + \frac{b}{a}\right)$

$$= (\varphi_4 \circ \varphi_3)\left(\frac{1}{\bar{z} + b/a}\right) = \varphi_4\left(\frac{1/a^2}{\bar{z} + b/a}\right)$$

$$= \varphi_4\left(\frac{1}{a^2\bar{z} + ab}\right) = \frac{1}{a^2\bar{z} + ab} + \frac{c}{a}$$

More isometries of $(\mathbb{H}^2, d_{hyp})$

$$(\varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1)(z) = \frac{1}{a^2\bar{z} + ab} + \frac{c}{a} = \frac{1}{a(a\bar{z}+b)} + \frac{c(a\bar{z}+b)}{a(a\bar{z}+b)}$$

$$= \frac{(ad - \cancel{bc}) + ac\bar{z} + \cancel{bc}}{a(a\bar{z}+b)}$$

$$= \frac{a(c\bar{z}+d)}{a(a\bar{z}+b)} = \frac{c\bar{z}+d}{a\bar{z}+b}.$$

Similarly, $(\varphi_2 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1)(z) = \dfrac{az+b}{cz+d}$.

Since both are compositions of isometries, they're isometries.

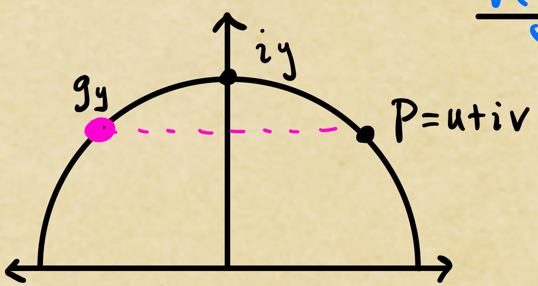
The case $a=0$ is an exercise. Hint: If $a=0$, then $ad-bc=1$ implies that $c \neq 0$. $\diamond$

All isometries of $(\mathbb{H}^2, d_{hyp})$

Our goal is to show that *all* isometries of $(\mathbb{H}^2, d_{hyp})$ have the above form. First we need:

<u>Lemma</u> Let φ be an isometry of $(\mathbb{H}^2, d_{hyp})$ s.t. $\varphi(iy) = iy$, for every $y > 0$. Then $\varphi(z) = z$ or $\varphi(z) = -\overline{z}$.

(Proof.) Given $P = u + iv \in \mathbb{H}^2$, let g_y be the complete hyperbolic geodesic thru P and intersecting $L = \{\operatorname{Re} z = 0\}$ perpendicularly. Let iy be $g_y \cap L$.



<u>Claim</u>. $\varphi(g_y) = g_y$.

By lemma on Jan 17, iy is the point on g_y nearest to iy_1, for any $y_1 > y_0$. $\therefore$ $\varphi(iy)$ is the point on $\varphi(g_y)$ nearest $\varphi(iy_1)$.

All isometries of $(\mathbb{H}^2, d_{hyp})$

But $\varphi(iy) = \underline{iy}$ and $\varphi(iy_1) = \underline{iy_1}$. Lemma $\Rightarrow$ $\underline{\varphi(g_y) \perp L}$.

$\therefore \varphi(g_y) = g_y$ ✓

Now $\varphi(P)$ is an element of $\underline{\varphi(g_y) = g_y}$ and

hypoth.

$$d(\varphi(P), iy) = d(\varphi(P), \varphi(iy)) = d(P, iy).$$

b/c φ is an isom.

$\therefore \varphi(P) = \pm u + iv$, i.e., $\varphi(P) = P$ or $\varphi(P) = -\overline{P}$.

This holds for any point in $\mathbb{H}^2$. Since φ is $\underline{\text{continuous}}$ and $\underline{\text{bijective}}$, we must have

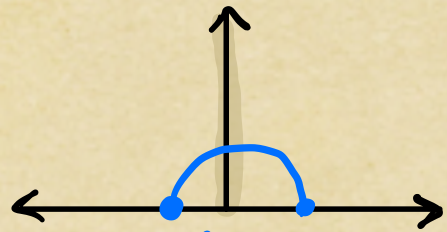
$$\varphi(z) = z, \quad \forall z \in \mathbb{H}^2$$

$$\text{or} \quad \varphi(z) = -\overline{z}, \quad \forall z \in \mathbb{H}^2.$$

◇

All isometries of $(\mathbb{H}^2, d_{hyp})$

Finally:



<u>Theorem</u>. If $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry of $(\mathbb{H}^2, d_{hyp})$, then there are $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ such that φ has the form

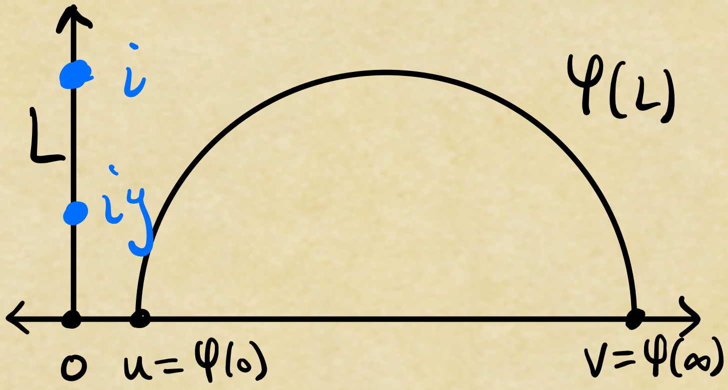
$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{or} \quad \varphi(z) = \frac{c\bar{z}+d}{a\bar{z}+b}.$$

(Proof.) Since $L = \{z \in \mathbb{H}^2 \mid \operatorname{Re} z = 0\}$ is a complete geodesic, so is $\varphi(L)$. We will consider the case where $\varphi(L)$ is a Euclidean semicircle with endpoints

$$u = \lim_{y \to 0} \varphi(iy) \quad \& \quad v = \lim_{y \to \infty} \varphi(iy).$$

The case where $\varphi(L)$ is a vertical line is an exercise.

All isometries of $(\mathbb{H}^2, d_{hyp})$



Now pick $a, c \in \mathbb{R}$ s.t. $ac(u-v)=1$ and consider

$$\psi(z) = \frac{a(z-u)}{c(z-v)}.$$

Then $\psi(u)=0$ and $\varphi(v)=\infty$, so $(\psi\circ\varphi)(0)=0$ & $(\psi\circ\varphi)(\infty)=\infty$. $\therefore (\psi\circ\varphi)(i)=it$ for some $t>0$.

We can replace a with $a/\sqrt{t}$ & c with $c\sqrt{t}$ in the def'n of ψ to ensure that $(\psi\circ\varphi)(i)=i$. $\therefore (\psi\circ\varphi)(iy)=iy$, $\forall\, y>0$.

Lemma $\Rightarrow (\psi\circ\varphi)(z)=z$ or $(\psi\circ\varphi)(z)=\overline{-z}$.

Case 1: $\varphi(z)=\psi^{-1}(z)=\dfrac{-cvz+au}{-cz+a}$

Case 2: $\varphi(z)=\psi^{-1}(-\bar{z})=\dfrac{cv\bar{z}+au}{c\bar{z}+a}$.

These have the desired form. $\diamond$

(Anti)linear fractional maps

A linear fractional map has the form

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ are s.t. φ is non constant.

Notice that $-d/c$ is not in the domain of φ. If $a, b, c, d \in \mathbb{R}$, we still get a map $\mathbb{H}^2 \to \mathbb{H}^2$. But it will be convenient to work with $a, b, c, d \in \mathbb{C}$, so we have some work to do.

We'll also consider antilinear fractional maps : non constant maps of the form $\varphi(z) = \dfrac{c\bar{z}+d}{a\bar{z}+b}$.

(Anti)linear fractional maps

To make sense of $\varphi(z)$ at $z = -d/c$, we'll consider φ as a map $\varphi : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \underline{\mathbb{C} \cup \{\infty\}}$ is the Riemann sphere. Namely, we will write $\varphi(z) = \dfrac{az+b}{cz+d}$ to mean

$$\varphi(z) = \begin{cases} \infty, & z = -d/c \\ a/c, & z = \infty \\ \dfrac{az+b}{cz+d}, & z \in \mathbb{C} - \{-d/c\} \end{cases}$$

(If $c = 0$, $-d/c$ means ∞.)

A similar approach works for antilinear fractional maps.

Exercises.

① Check that (A)LFM nonconstant $\Rightarrow$ we can take $ad - bc = 1$.

② Determine which of the following form groups under composition:

$$\{\text{LFMs}\}, \quad \{\text{ALFMs}\}, \quad \{\text{(A)LFMs}\}.$$

③ Watch "Mobius Transformations Revealed" on YouTube.

https://youtu.be/JX3VmDgiFnY?si=JFoPTC0PmxZ8YqGY

Our favorite (A)LFMs

Some familiar maps are (A)LFMs:

homotheties: $z \mapsto \lambda z = \dfrac{\lambda z + 0}{0z + 1} = \dfrac{\sqrt{\lambda}\, z + 0}{0\, z + 1/\sqrt{\lambda}}$

translations: $z \mapsto z + z_0 = \dfrac{z + z_0}{0z + 1}$

Note: We're allowing $z_0 \notin \mathbb{R}$

inversion across $|z| = 1$: $z \mapsto \dfrac{1}{\bar{z}} = \dfrac{0\bar{z} + 1}{\bar{z} + 0}$

We also now have rotations:

$$z \mapsto e^{i\theta} z = \frac{e^{i\theta} z + 0}{0z + 1}$$

Lemma. Every (A)LFM is a composition of maps of the above type.