Math 4803 January 17, 2024

LAST TIME

① The hyperbolic plane as a metric space.

② A few fundamental isometries:
horizontal translations, homotheties, & the standard inversion.

TODAY

① Shortest curves in $(\mathbb{H}^2, d_{hyp})$.

~~② More isometries of $(\mathbb{H}^2, d_{hyp})$.~~

Shortest curves in $(\mathbb{H}^2, d_{hyp})$

Recall:

Prop. The line segment $[P, Q]$ minimizes Euclidean length among p.w.d. curves from P to Q.

Swithching from $\ell_{euc}(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ to $\ell_{hyp}(\gamma) = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\, dt$ will change our characterization of length-minimizing curves.

Shortest curves in $(\mathbb{H}^2, d_{hyp})$

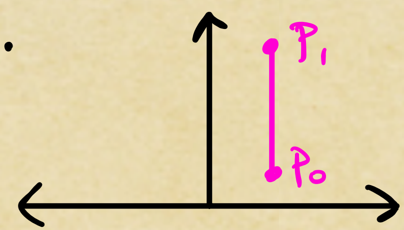
Prop. If $P_0 = (x_0, y_0)$, $P_1 = (x_0, y_1) \in \mathbb{H}^2$ are located on the same vertical line, then

① the line segment $[P_0, P_1]$ has the shortest hyperbolic length among all p.w.d. curves $P_0 \rightsquigarrow P_1$;

② the hyperbolic length of any other p.w.d. curve $P_0 \rightsquigarrow P_1$ is strictly greater;

③ $d_{hyp}(P_0, P_1) = \ell_{hyp}([P_0, P_1]) = \left| \ln y_1/y_0 \right|$.

typo in book!

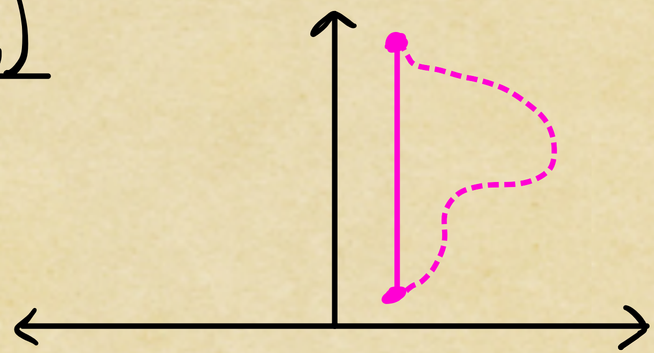


(Proof.) First, let's compute $\ell_{hyp}([P_0, P_1])$:

$t \mapsto (x_0, t)$, $y_0 \le t \le y_1$ (WLOG, $y_1 \ge y_0$)

$$\ell_{hyp}([P_0,P_1]) = \int_{y_0}^{y_1} \frac{\sqrt{0^2 + 1^2}}{t}\, dt = \int_{y_0}^{y_1} \frac{1}{t}\, dt = \ln t \Big|_{y_0}^{y_1} = \ln \frac{y_1}{y_0}.$$

Shortest curves in $(\mathbb{H}^2, d_{hyp})$



Next, consider any $P_0 \overset{\gamma}{\rightsquigarrow} P_1$:

$t \mapsto (x(t), y(t)),\quad a \le t \le b.$

Then $$\ell_{hyp}(\gamma) = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\,dt$$

$$\geq \int_a^b \frac{\sqrt{0^2 + (y'(t))^2}}{y(t)}\,dt = \int_a^b \frac{|y'(t)|}{y(t)}\,dt$$

$$\geq \int_a^b \frac{y'(t)}{y(t)}\,dt = \ln y(t)\Big|_a^b = \ln \frac{y(b)}{y(a)} = \ln \frac{y_1}{y_0}.$$

The first $\geq$ achieves $=$ iff $x(t) \equiv x_0$, while the second $\geq$ achieves $=$ iff $y'(t) \geq 0$.

$\diamond$

Shortest curves in $(\mathbb{H}^2, d_{hyp})$

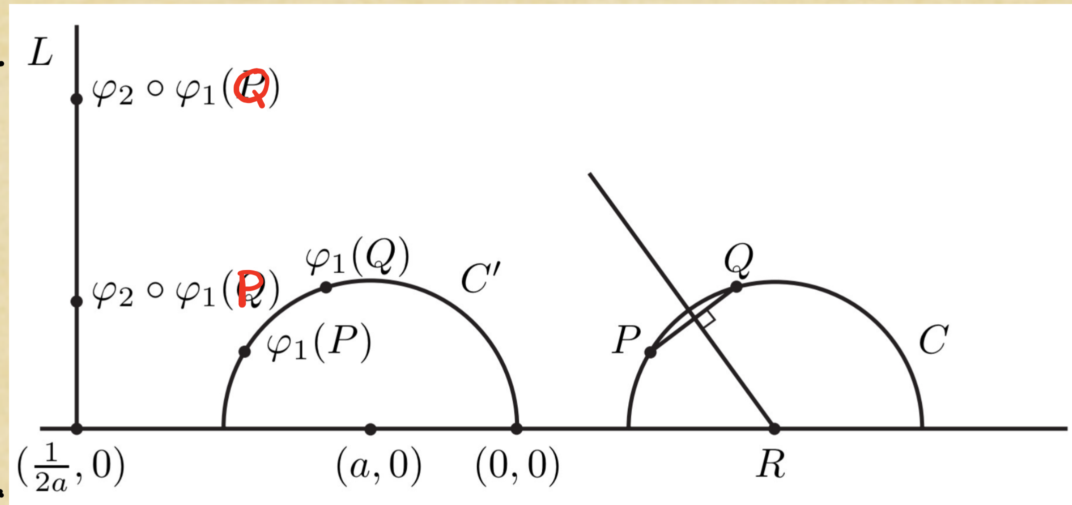
Next, we need to consider points with distinct x-coords.

<u>Lemma for later</u> For any $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1) \in \mathbb{H}^2$,

$$d_{hyp}(P_0, P_1) \geq \left| \ln y_1/y_0 \right|$$

(Proof.) Exercise w/ previous proof technique. $\diamond$

<u>Lemma</u>. For any $P, Q \in \mathbb{H}^2$ with $x(P) \neq x(Q)$, $\exists$ <u>isometry</u> $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ s.t. <u>$x(\varphi(P)) = x(\varphi(Q))$</u>.



Under φ, the <u>circular arc</u> joining P to Q and <u>is centered on the x-axis</u> is mapped to <u>$[\varphi(P), \varphi(Q)]$</u>.

Shortest curves in $(\mathbb{H}^2, d_{hyp})$

Lemma. For any $P, Q \in \mathbb{H}^2$ with $x(P) \neq x(Q)$, $\exists$ isometry $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ s.t. $x(\varphi(P)) = x(\varphi(Q))$. Under φ, the circular arc joining P to Q and centered on the x-axis is mapped to $[\varphi(P), \varphi(Q)]$.

(Proof.) Let C = circle <u>passing thru P & Q with center on the x-axis</u>. Let $\varphi_1 : \mathbb{H}^2 \to \mathbb{H}^2$ be a <u>horizontal translation</u> ensuring that $C' = \varphi_1(C)$ <u>passes thru $(0,0)$</u>.

In polar coordinates, C' has equation <u>$r = 2a\cos\theta$</u>, for some $a \in \mathbb{R}$. If φ_2 is the standard inversion, then $(\varphi_2 \circ \varphi_1)(C) = \varphi_2(C')$ has polar eqn <u>$r = \frac{1}{2a\cos\theta}$</u>.

i.e., $r\cos\theta = \frac{1}{2a} \to x = \frac{1}{2a}$, a vertical line $\diamond$

Shortest curves in $(\mathbb{H}^2, d_{hyp})$

We can now describe all shortest curves.

<u>Thm</u>. Fix $P, Q \in \mathbb{H}^2$. The unique p.w.d. curve joining P to Q in $\mathbb{H}^2$ which minimizes $\ell_{hyp}(\gamma)$ is the <u>circular arc $P \rightsquigarrow Q$ centered on the x-axis</u> (possibly a <u>vertical line</u>) passing thru P & Q.

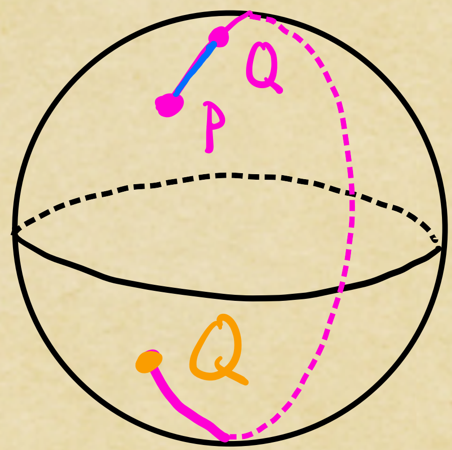
(Proof.) If $x(P) = x(Q)$, see first proposition. If $x(P) \neq x(Q)$, previous lemma gives an isometry s.t. $x(\varphi(P)) = x(\varphi(Q))$ and the described arc is mapped to $[\varphi(P), \varphi(Q)]$. Since isometries preserve shortest curves, the arc is shortest.

$\diamond$

Shortest curves in $(\mathbb{H}^2, d_{hyp})$

<u>Thm</u>. Fix $P, Q \in \mathbb{H}^2$. The unique p.w.d. curve joining P to Q in $\mathbb{H}^2$ which minimizes $\ell_{hyp}(\gamma)$ is the circular arc centered on the x-axis (possibly a vertical line) passing thru P & Q.

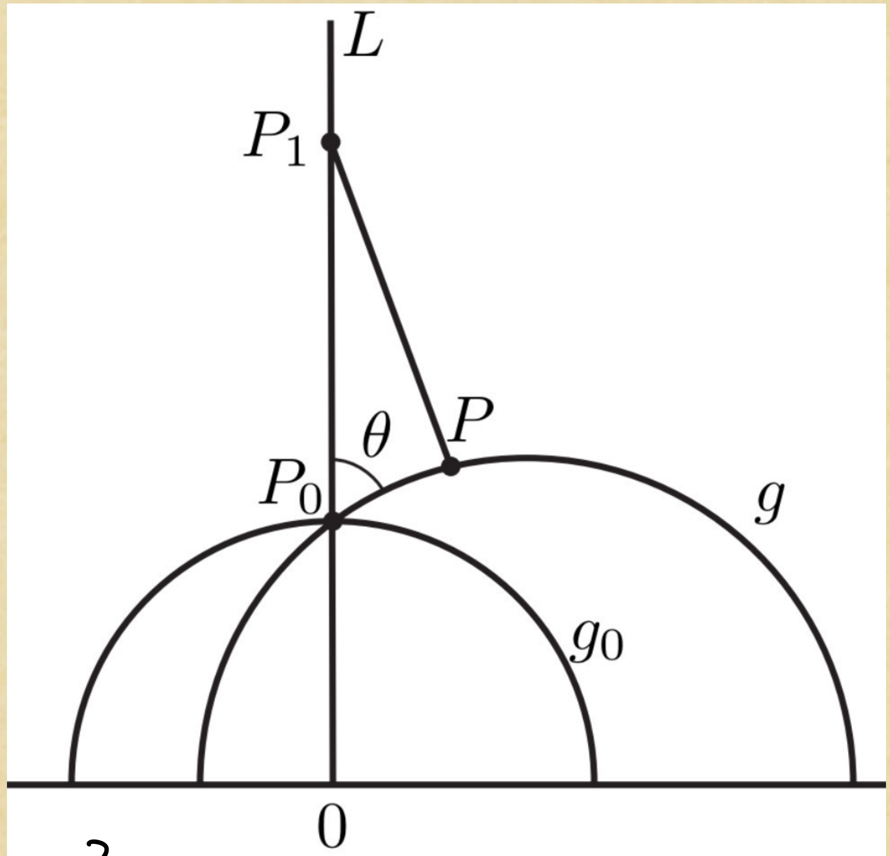
A **geodesic** is a curve γ s.t., for every $P \in \gamma$ and every $Q \in \gamma$ <u>suff. close to P</u>, the arc $P \rightsquigarrow Q$ is the <u>shortest curve</u> $P \rightsquigarrow Q$.



A <u>**complete geodesic**</u> is a geodesic which cannot be extended any further. (line vs. line segment)

Connecting points to geodesics

We'll need the following when studying isometries.



Lemma. Let $P_0 = (0, y_0)$ and $P_1 = (0, y_1)$, with $y_1 > y_0$, and let g be a complete hyperbolic geodesic passing thru P_0.

TFAE:

① P_0 is the point on g which is nearest P_1 w.r.t. d_{hyp};

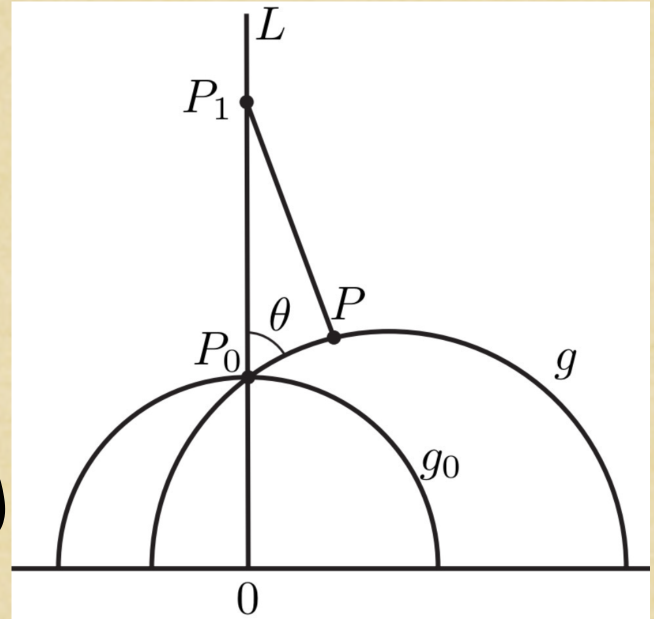
② g is perpendicular $L = \{x=0\}$ at P_0.

i.e., g is the Euclidean semi-circle of radius y_0 joining $(y_0, 0)$ to $(-y_0, 0)$.

Connecting points to geodesics

We'll prove (1) $\Rightarrow$ (2).

§ g makes an angle $\theta < \frac{\pi}{2}$ with L as it passes thru P_0.



NTS: $\exists\, P \in g$ s.t. $d_{hyp}(P_1, P) < d_{hyp}(P_1, P_0)$

Group work! Write $P = (u, v) \in g$.

Then $[P_1, P]$ is param. by $t \mapsto [ut, t(v-y_1)+y_1]$, $0 \le t \le 1$.

$$\ell_{hyp}([P_1, P]) = \int_0^1 \frac{\sqrt{u^2 + (v-y_1)^2}}{t(v-y_1)+y_1}\, dt = \frac{\sqrt{u^2+(v-y_1)^2}}{v-y_1} \int_0^1 \frac{dt}{t + \frac{y_1}{v-y_1}}$$

$$= \frac{\sqrt{u^2+(v-y_1)^2}}{v-y_1}\left[\ln\left|1 + \frac{y_1}{v-y_1}\right| - \ln\left|0 + \frac{y_1}{v-y_1}\right|\right]$$

Connecting points to geodesics

$$\ell_{hyp}([P_1,P]) = \frac{\sqrt{u^2+(v-y_1)^2}}{v-y_1} \cdot \ln \left| \frac{1+\frac{y_1}{v-y_1}}{\frac{y_1}{v-y_1}} \right|$$

$$= \frac{\sqrt{u^2+(v-y_1)^2}}{v-y_1} \cdot \ln \left| \frac{(v-y_1)+y_1}{y_1} \right| = \pm\sqrt{\left(\frac{u}{v-y_1}\right)^2+1}\ \ln\left|\frac{v}{y_1}\right|.$$

Now we want $\frac{d}{du}\left(\ell_{hyp}(P_1,P)\right)$ @ $u=0$.

$$\frac{d}{du}\left(\ell_{hyp}(P_1,P)\right) = \frac{d}{du}\left(\pm\sqrt{\left(\frac{u}{v-y_1}\right)^2+1}\cdot\ln\left|\frac{v}{y_1}\right|\right)$$

$$= \pm\frac{1}{2}\left(\left(\frac{u}{v-y_1}\right)^2+1\right)^{-1/2}\cdot\left(2\left(\frac{u}{v-y_1}\right)\cdot\left(\frac{v-y_1-u\cdot\frac{dv}{du}}{(v-y_1)^2}\right)\right)\cdot\ln\left|\frac{v}{y_1}\right|$$

$$\pm\sqrt{\left(\frac{u}{v-y_1}\right)^2+1}\cdot\frac{\frac{dv}{du}}{v}.$$

$u=0 \Rightarrow v=y_0$; $\frac{dv}{du}=\cot\theta$

$$\frac{d}{du}\left(\ell_{hyp}(P_1,P)\right)\bigg|_{u=0} = 0-\sqrt{0+1}\cdot\frac{\cot\theta}{y_0} = -\frac{\cot\theta}{y_0}$$

Near $u=0$, $v-y_1<0$

Connecting points to geodesics

So $\dfrac{d}{du}\left(\ell_{hyp}(P_1, P)\right)\Big|_{u=0} = -\dfrac{\cot\theta}{y_0} \neq 0$, if $\theta \neq \pi/2$.

Upshot: $\exists\, P \in g$ near P_0 with $\ell_{hyp}(P_1, P) < \ell_{hyp}(P_1, P_0)$.

But then we have

$$d_{hyp}(P_1, P) \leq \ell_{hyp}(P_1, P) < \ell_{hyp}(P_1, P_0) = d_{hyp}(P_1, P_0).$$

So $\theta \neq \frac{\pi}{2} \Rightarrow P_0$ is not the point on g nearest P_1.

$\diamond$