Math 4803 January 10, 2024

LAST TIME

We studied the geometry of the Euclidean plane as a metric space. Metric-preserving bijections were called isometry.

TODAY

1. The hyperbolic plane as a metric space.
2. A few fundamental isometries.
3. A crash course in complex numbers.

The hyperbolic plane

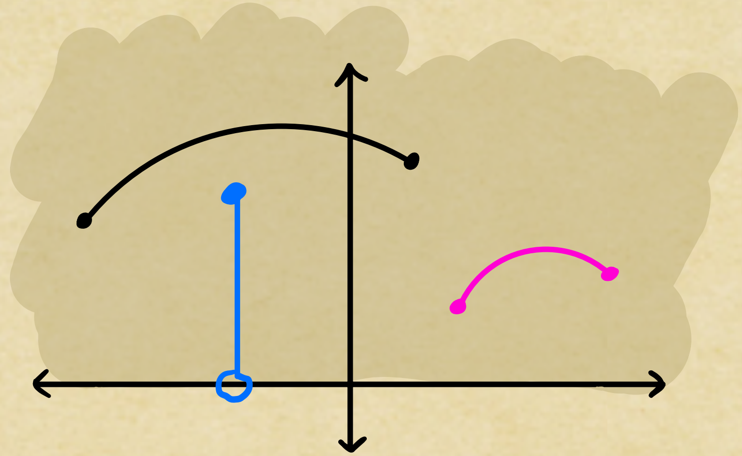
As a set, the hyperbolic plane is given by

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Lengths are measured very differently:

If γ is $[a,b] \longrightarrow \mathbb{H}^2$

$t \longmapsto (x(t), y(t))$,



then

$$\ell_{hyp}(\gamma) := \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\, dt.$$

(We assume that γ is piecewise diff'able.)

Hyperbolic distance

Once we've defined lengths, distance is familiar:

$$d_{hyp}(P,Q) := \inf\left\{ \ell_{hyp}(\gamma) \;\middle|\; \begin{array}{l}\gamma \text{ is a p.w.d. curve} \\ \text{in } \mathbb{H}^2 \text{ from } P \text{ to } Q\end{array} \right\}$$

i.e., d_{hyp} is a path metric

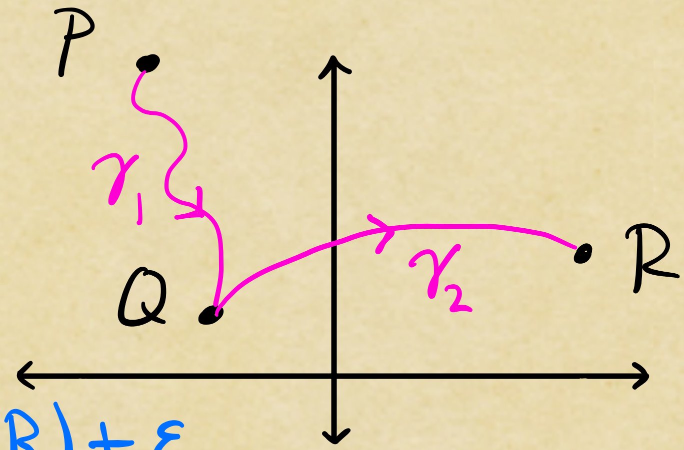
Prop. The function d_{hyp} is a metric.

(Proof.)

$d_{hyp} \geq 0$? $\quad \dfrac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)} \geq 0$ ✓

symmetric? Any curve from P to Q can be reoriented as a curve from Q to P without changing its length. ✓

Hyperbolic distance, continued



triangle inequality?

Given $\varepsilon > 0$, we want a p.w.d. curve γ s.t. $P \overset{\gamma}{\rightsquigarrow} R$ & $\ell_{hyp}(\gamma) \leq d_{hyp}(P,Q) + d_{hyp}(Q,R) + \varepsilon$.

By def'n of d_{hyp}, $\exists$ γ_1 and γ_2 s.t.

$\ell_{hyp}(\gamma_1) \leq d_{hyp}(P,Q) + \frac{1}{2}\varepsilon$ & $\ell_{hyp}(\gamma_2) \leq d_{hyp}(Q,R) + \frac{1}{2}\varepsilon$,

so we can let γ be the concatenation of γ_1 & γ_2.

Then
$$\begin{aligned} \ell_{hyp}(\gamma) &= \ell_{hyp}(\gamma_1) + \ell_{hyp}(\gamma_2) \\ &\leq d_{hyp}(P,Q) + d_{hyp}(Q,R) + \varepsilon. \end{aligned}$$

Since this holds $\forall\, \varepsilon > 0$, $d_{hyp}(P,R) \leq d_{hyp}(P,Q) + d_{hyp}(Q,R)$.

Hyperbolic distance, continued

$$\boxed{\ell_{hyp}(\gamma) = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\,dt}$$

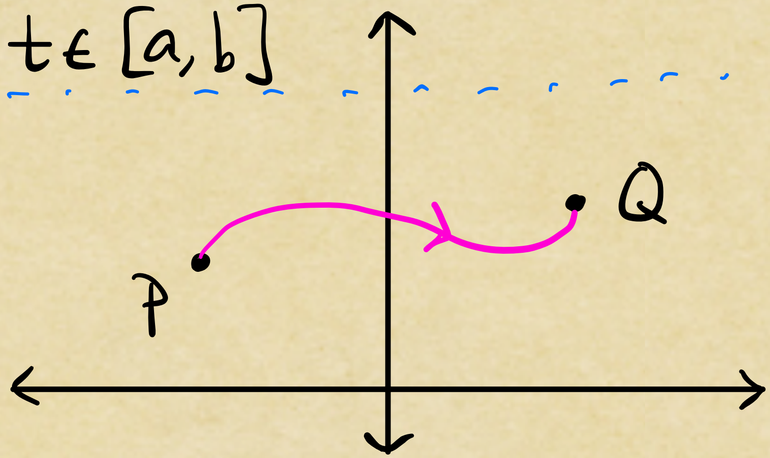
$d_{hyp}(P,Q) = 0$ iff $P = Q$?

If $P = Q$, then let γ be constant.

If $P \neq Q$, we need to find a lower bound $C > 0$ s.t. every p.w.d. curve γ from P to Q has $\ell_{hyp}(\gamma) \geq C$.

Say we have $t \mapsto (x(t), y(t))$, with $P = (x(a), y(a))$, $Q = (x(b), y(b))$.

Case 1: $y(t) \leq 2y(a)$, $\forall\, t \in [a,b]$

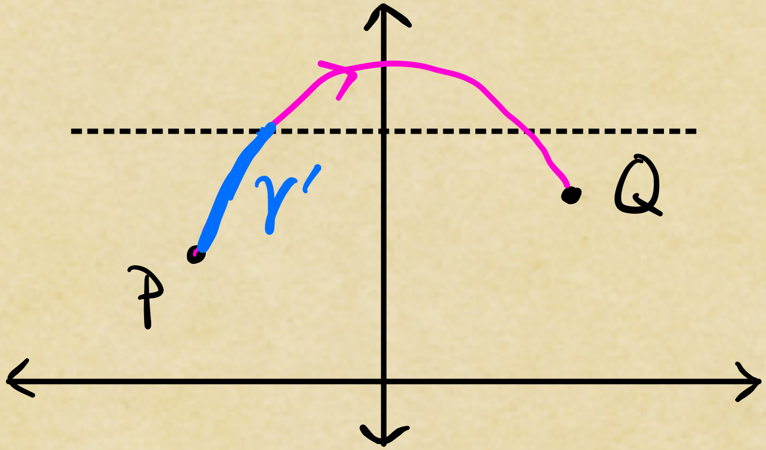


$$\ell_{hyp}(\gamma) = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\,dt$$

$$\geq \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{2y(a)}\,dt$$

$$= \frac{1}{2y(a)}\,\ell_{euc}(\gamma) \geq \frac{1}{2y(a)}\,d_{euc}(P,Q) > 0.$$

Hyperbolic distance, continued



Case ②: γ crosses $y = 2y(a)$

Let γ' be the arc of γ which connects P to $\{y = 2y(a)\}$.

By Case ①, $\ell_{hyp}(\gamma') \geq \frac{1}{2y(a)} \ell_{euc}(\gamma') \geq \frac{1}{2y(a)} \cdot \underbrace{y(a)}_{\text{Euclidean dist. from P to the line}}$

So $\ell_{hyp}(\gamma) \geq \ell_{hyp}(\gamma') \geq \frac{1}{2}$.

In either case, $\ell_{hyp}(\gamma) \geq C = \min\left\{\frac{d_{euc}(P,Q)}{2y(a)}, \frac{1}{2}\right\} > 0$,

So taking the infimum over all curves yields

$$d_{hyp}(P,Q) \geq \min\left\{\frac{d_{euc}(P,Q)}{2y(a)}, \frac{1}{2}\right\} > 0. \qquad \diamond$$

First isometries

$$\ell_{hyp}(\gamma) = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\,dt$$

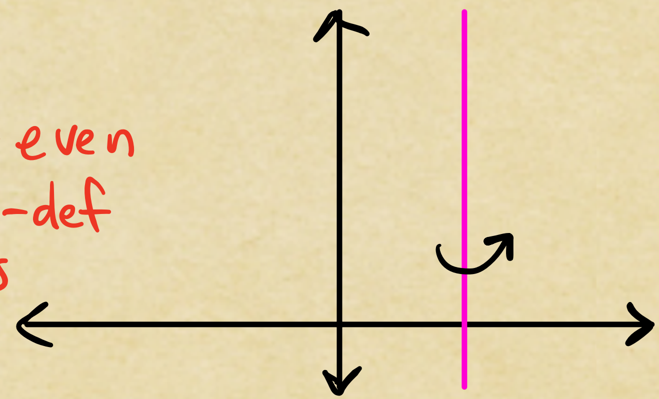
Here are some (non-) isometries of $(\mathbb{H}^2, d_{hyp})$:

horizontal translation $(x,y) \mapsto (x + x_0, y)$

NOT vertical translation $(x,y) \longmapsto (x, y + y_0)$

adding x_0 to x doesn't change x', y', or y

Not even well-def maps



reflection over vertical lines

$(x,y) \mapsto (2y_0 - x, y)$

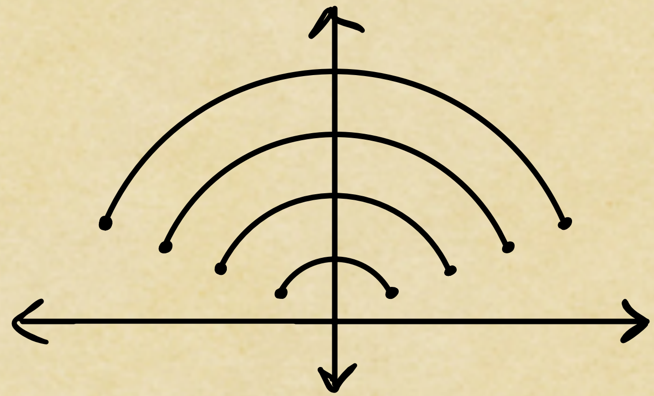
NOT reflection over horizontal lines

doesn't change $(x')^2$, y', or y

First isometries

$$\boxed{\ell_{hyp}(\gamma) = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\, dt}$$

Our first counterintuitive isometry is the homothety.



For any $\lambda > 0$, we can define

$$\varphi(x,y) = (\lambda x, \lambda y).$$

Then
$$\ell_{hyp}(\varphi(\gamma)) = \int_a^b \frac{\sqrt{(\lambda x'(t))^2 + (\lambda y'(t))^2}}{\lambda y(t)}\, dt$$
$$= \frac{|\lambda|}{\lambda} \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\, dt = 1 \cdot \ell_{hyp}(\gamma).$$

Since this holds for all γ, we have

$$d_{hyp}(\varphi(P), \varphi(Q)) = d_{hyp}(P, Q), \quad \forall\, P, Q \in \mathbb{H}^2.$$

Homogeneity

Prop. The hyperbolic plane $(\mathbb{H}^2, d_{hyp})$ is homogeneous.

i.e., $\forall\ P, Q \in \mathbb{H}^2$, $\exists$ an isom. $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ s.t. $\varphi(P) = Q$.

(Proof.) Group work! $P = (a, b)$ & $Q = (c, d)$

Let $\varphi_1(x,y) = \left(\frac{d}{b}x, \frac{d}{b}y\right)$ (to get y-values to match).

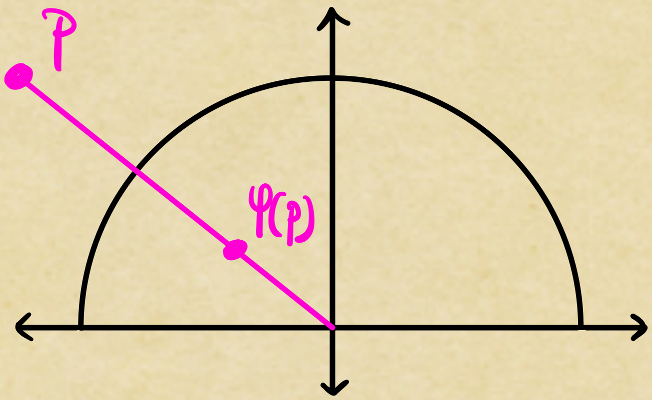
Then $\varphi_1(a,b) = \left(\frac{ad}{b}, d\right)$. Let $\varphi_2(x,y) = \left(x + \left(c - \frac{ad}{b}\right), y\right)$.

Then $\varphi_2\left(\frac{ad}{b}, d\right) = (c, d)$. So $(\varphi_2 \circ \varphi_1)(a,b) = (c,d)$.

Let $\varphi = \varphi_2 \circ \varphi_1$.

$\diamond$

Standard inversion



Prop. The standard inversion

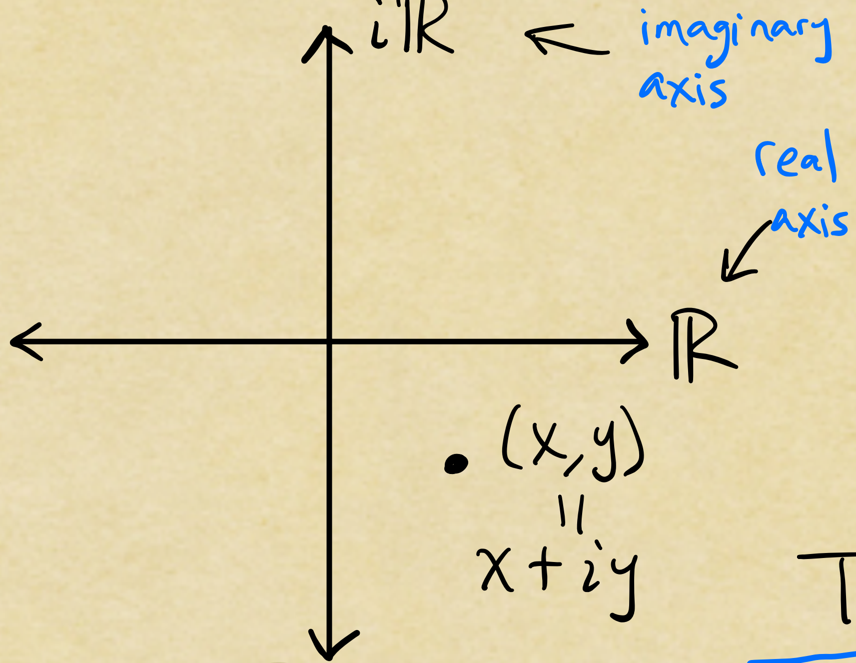
$$\varphi(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

(or $\varphi(z) = \frac{1}{\bar{z}}$) is an isometry of $(\mathbb{H}^2, d_{hyp})$.

(Proof.) See pp. 16-17 of book.

Ran out of time :(

Complex numbers crash course



$z = x + iy$

$\Rightarrow \operatorname{Re} z = x$ & $\operatorname{Im} z = y$

The **complex conjugate** is $\overline{z} = x - iy$.

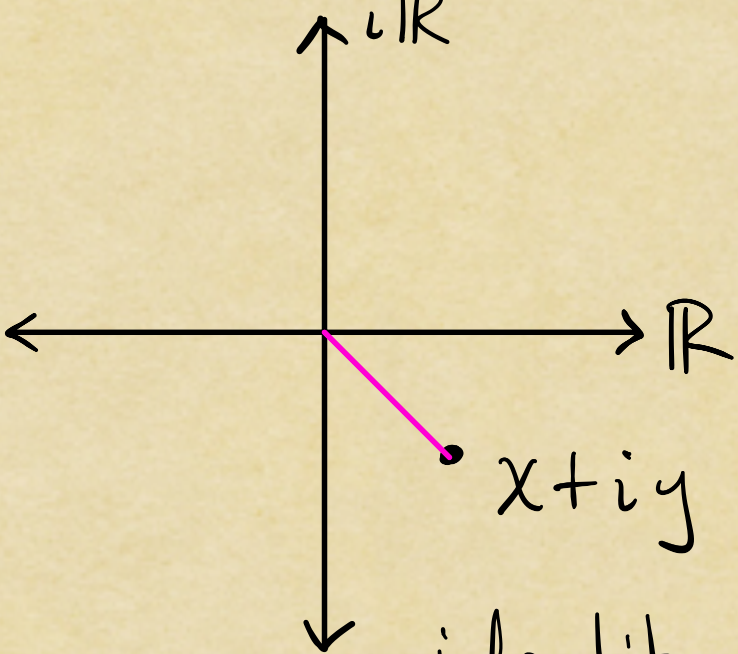
The **modulus** or **absolute value** is $|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}$.

These behave nicely: for any $z, w \in \mathbb{C}$,

$\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z} \cdot \overline{w}$, & $|zw| = |z| \cdot |w|$.

(But $|z+w| \neq |z| + |w|$!)

Complex numbers crash course



Helpful formulas:

$\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$ & $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$.

We often prefer to write z in polar form, using the identity $e^{i\theta} = \cos\theta + i\cdot\sin\theta$.

$z = re^{i\theta} \Rightarrow \operatorname{Re} z = r\cos\theta$, $\operatorname{Im} z = r\sin\theta$, & $|z| = r$.

In fact, we can compute e^z for any $z = x+iy$:

$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\cdot\sin y)$.

Complex numbers crash course

Ex ① $\dfrac{4+i}{6-3i} = \dfrac{4+i}{6-3i}\cdot\dfrac{6+3i}{6+3i} = \dfrac{24-3+12i+6i}{36+9}$

$= \boxed{\dfrac{21}{45} + \dfrac{18}{45}i}$

② $\left|\dfrac{3-i}{(6+2i)^3}\right| = \dfrac{|3-i|}{|6+2i|^3} = \dfrac{(9+1)^{1/2}}{(36+4)^{3/2}} = \boxed{\sqrt{\dfrac{10}{40^3}}}$

③ $\sqrt{3}+i$ in polar: $r = |\sqrt{3}+i| = (3+1)^{1/2} = 2$

$\sqrt{3}+i = 2(\cos\theta + i\sin\theta) \rightarrow \theta = \frac{\pi}{6} \rightarrow \sqrt{3}+i = \boxed{2e^{i\pi/6}}$

④ $3e^{i\pi} = 3(\cos\pi + i\cdot\sin\pi) = 3(-1+i\cdot 0)$

$= \boxed{-3}$