# Homework 3

> **Instructions**
>
> Complete the exercises on this page and upload your work to Gradescope by **12:29pm on February 19**.
>
> Be sure to **acknowledge your collaborators**.

## Exercises

1. Textbook exercise 4.5

2. Consider the partition $\bar{X}$ of the plane $X = \mathbb{R}^2$ consisting of all subsets $A \subset X$ of the form

$$A = \{(x, y), (-x, y)\},$$

   for some $x, y \in \mathbb{R}$. Note that if $x = 0$, then $A$ contains a single element. Let $\bar{d}_{\text{euc}}$ be the quotient semi-metric on $\bar{X}$ induced by the Euclidean metric $d_{\text{euc}}$ of $X = \mathbb{R}^2$. We will analyze $(\bar{X}, \bar{d}_{\text{euc}})$.

   (a) Show that $\bar{X}$ is a partition of $X = \mathbb{R}^2$.

   (b) Let $X_+ = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$ and consider the map $\varphi : X_+ \to \bar{X}$ which associates to each point $P \in X_+$ the subset $\varphi(P) = \bar{P} \in \bar{X}$ which contains it. Show that $\varphi$ is bijective.

   (c) For any $P = (x, y)$ and $Q = (u, v)$ in $\mathbb{R}^2$, let $P' = (|x|, y)$ and $Q' = (|u|, v)$ in $X_+$. Show that $d_{\text{euc}}(P', Q') \leq d_{\text{euc}}(P, Q)$.

   (d) For $P, Q \in X_+$, consider a discrete walk $w$ from $\bar{P}$ to $\bar{Q}$ consisting of the points

   $$P = P_1, Q_1 \sim P_2, Q_2 \sim P_3, \ldots, Q_{n-1} \sim P_n, Q_n = Q$$

   of $X$. Show that there is another discrete walk $w'$ consisting of points

   $$P = P_1', Q_1' = P_2', Q_2' = P_3', \ldots, Q_{n-1}' = P_n', Q_n' = Q$$

   such that $\ell_{d_{\text{euc}}}(w') \leq \ell_{d_{\text{euc}}}(w)$.
   *Note: The use of $=$ rather than $\sim$ in $w'$ is not a typo. Part (c) should be helpful.*

   (e) For the discrete walk $w'$ of part (d), show that $\ell_{d_{\text{euc}}}(w') \geq d_{\text{euc}}(P, Q)$.

   (f) Show that $\bar{d}_{\text{euc}}(\bar{P}, \bar{Q}) = d_{\text{euc}}(P, Q)$ for every $P, Q \in X_+$. Conclude that the map $\varphi : X_+ \to X$ constructed above is an isometry from $(X_+, d_{\text{euc}})$ to $(\bar{X}, \bar{d}_{\text{euc}})$.

3. Textbook exercise 4.7. (You may use the results of exercises 4.4 and 4.5 without actually doing them.)

4. Textbook exercise 5.1.

5. Textbook exercise 5.3.

6. Textbook exercise 5.4.