Homework 1

Instructions

Complete the exercises on this page and upload your work to Gradescope by **12:29pm on January 22**. The optional exercises on page 2 need not be submitted.

Be sure to **acknowledge your collaborators**.

Exercises

1. Textbook exercise 1.2.
2. Textbook exercise 1.5.
3. Textbook exercise 1.10.
4. Textbook exercise 2.1.
5. Textbook exercise 2.2.
6. Textbook exercise 2.8.
7. Let $\varphi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be the map defined by the property that $\varphi(x,y) = (-x,y)$. (So φ is the Euclidean reflection across the y-axis.)
   (a) Show that if γ is a curve in $\mathbb{H}^2$, then $\ell_{\text{hyp}}(\varphi(\gamma)) = \ell_{\text{hyp}}(\gamma)$.
   (b) Use Part (a) to show that φ is an isometry from $(\mathbb{H}^2, d_{\text{hyp}})$ to itself.
8. Let
   $$\varphi(z) = \frac{az+b}{cz+d}, \quad \text{with} \quad a,b,c,d \in \mathbb{R}, ad-bc = 1, \text{ and } a \neq 0.$$
   Set
   $$\varphi_1(z) := z + \frac{b}{a}, \quad \varphi_2(z) := \frac{1}{\bar{z}}, \quad \varphi_3(z) := \frac{1}{a^2}z, \quad \varphi_4(z) := z + \frac{c}{a}.$$
   (a) Which of $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are horizontal translations, homotheties, or inversions?
   (b) Show that $\varphi = \varphi_2 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$.
   (c) Conclude that φ is an isometry of $(\mathbb{H}^2, d_{\text{hyp}})$.

Optional Exercises

1. Let $z = 2 - i$ and $w = 3 + 4i$. Write the product zw and the quotient $\frac{z}{w}$ in the form $a + i\,b$, with $a, b \in \mathbb{R}$.
2. Find $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$ so that $i - 1 = r\,e^{i\theta}$.
3. Let X be the plane $\mathbb{R}^2$, and let $d_1, d_2 \colon X \times X \to \mathbb{R}$ be defined by

$$d_1((x,y),(x',y')) := |x - x'| + |y - y'| \quad \text{and} \quad d_2((x,y),(x',y')) := \max\{|x - x'|, |y - y'|\}.$$

   Show that (X, d_1) and (X, d_2) are metric spaces.

4. Textbook exercise 1.11.
5. Let $\varphi_{z_0} \colon \mathbb{C} \to \mathbb{C}$ be the rotation of angle θ around the fixed point $z_0 \in \mathbb{C}$. Express $\varphi(z)$ in terms of z, z_0, and $e^{i\theta}$.
6. Describe in words the transformation $\psi \colon \mathbb{C} \to \mathbb{C}$ defined by $\psi(z) = -\overline{z}$.
7. Textbook exercise 2.6.
8. In the hyperbolic plane $\mathbb{H}^2$, consider the two points $P = i$ and $Q = 4 + i$. For $u > 0$, let $P_u = u\,i$, let $Q_u = 4 + u\,i$, and let γ_u be the curve going from P to Q that is made up of the vertical line segment $[P, P_u]$, followed by the horizontal line segment $[P_u, Q_u]$, and finally followed by the vertical segment $[Q_u, Q]$.
   (a) Draw a picture of γ_u.
   (b) Compute the hyperbolic length $\ell_{\text{hyp}}(\gamma_u)$.
   (c) For which value of u is $\ell_{\text{hyp}}(\gamma_u)$ minimum?
   (d) Show that $d_{\text{hyp}}(P, Q) \leq 2\ln 2 + 2$.