**Math 4803** **February 7, 2024**

RECENTLY

Quotients of metric spaces, which are generally <u>Semi-metric</u> spaces.

i.e., $\bar{d}(\bar{P},\bar{Q})=0 \not\Rightarrow \bar{P}=\bar{Q}$

TODAY

Quotients of <u>polygons</u> via <u>edge gluings</u>.

# Euclidean polygons

A polygon in $(\mathbb{R}^2, d_{euc})$ is a region $X \subset \mathbb{R}^2$ whose bdry decomposes as $\partial X = E_1 \cup E_2 \cup \cdots \cup E_n$, where

(1) each $E_i$ is a line segment, line, or half-line;

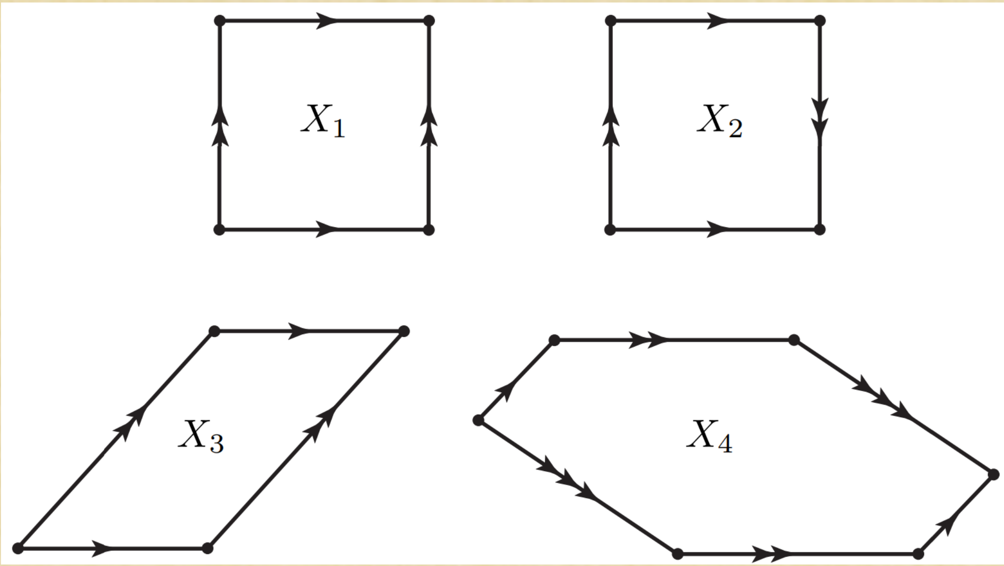
(2) each nonempty $E_i \cap E_j$ is an endpoint of $E_i$ & $E_j$, for $i \neq j$;

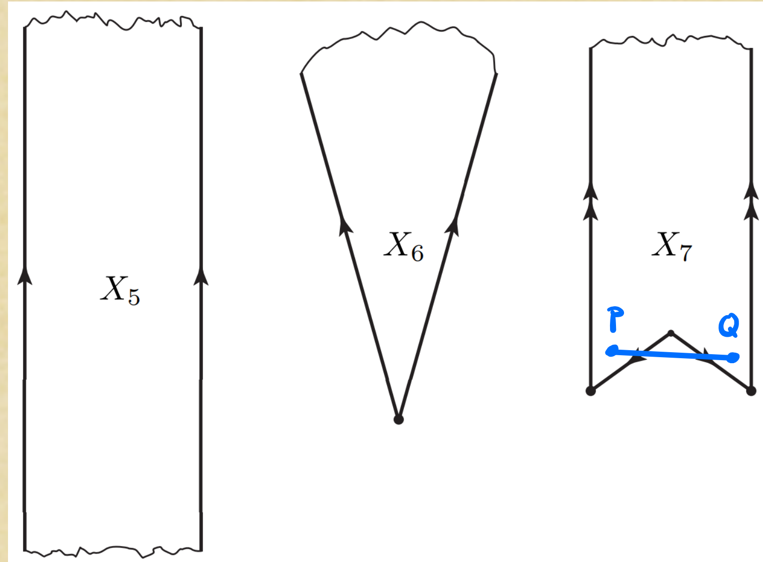
(3) for distinct $1 \leq i, j, k \leq n$, $E_i \cap E_j \cap E_k = \emptyset$.

We call each $E_i$ an edge of $X$ and call the nonempty $E_i \cap E_j$ vertices.

We require $X$ to be closed, meaning that $\partial X \subset X$, but do not require $X$ to be bounded.

# Euclidean polygons



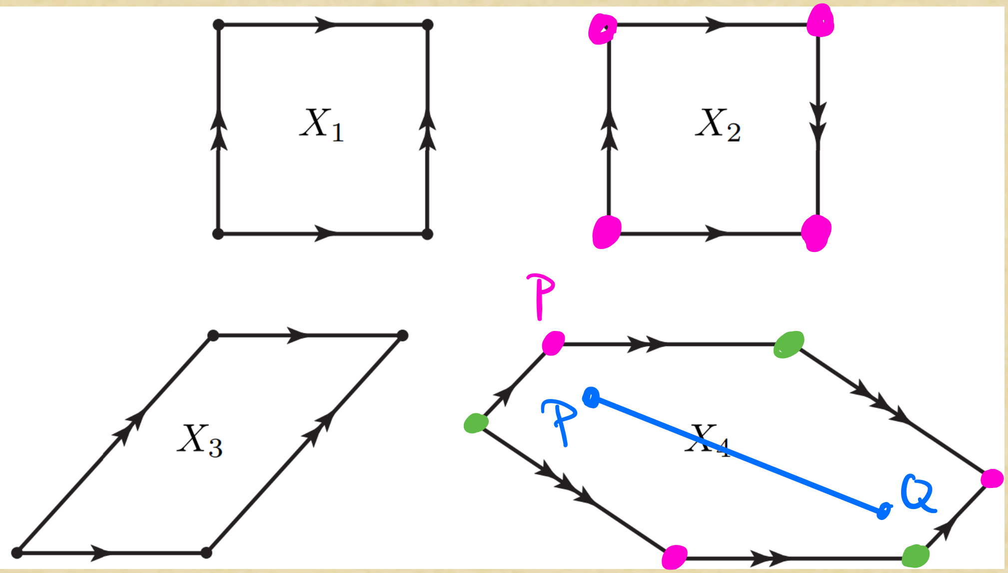



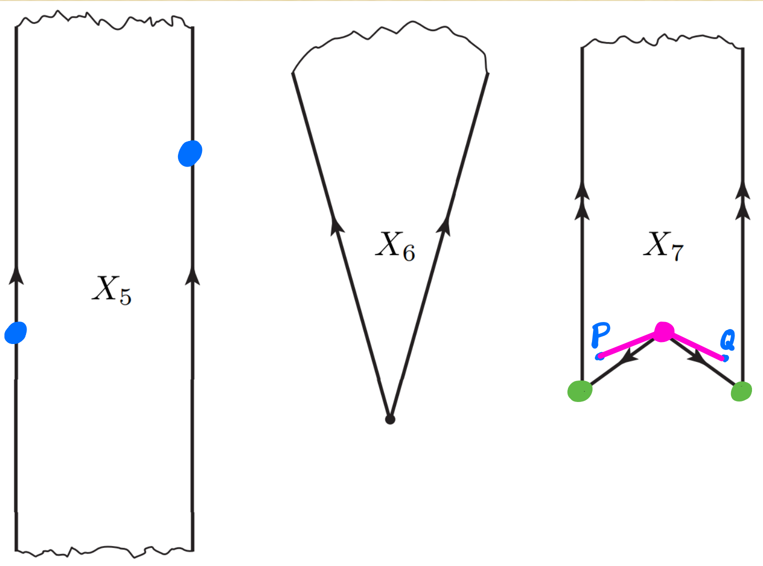

We call $X$ bounded if $\exists\ P \in \mathbb{R}^2$ and $r > 0$ s.t.

$$X \subset B_{d_{euc}}(P, r)$$

We call $X$ convex if, $\forall\ P, Q \in X$, there is a geodesic arc contained in $X$ connecting $P$ to $Q$.

# Euclidean polygons




The Euclidean path metric on X is defined by

$$d_X(P,Q) := \inf\{\ell_{euc}(\gamma) \mid P \overset{\gamma}{\rightsquigarrow} Q \text{ in } X\},$$

for all $P, Q \in X$.

Check: If X is convex, then

$$d_X \equiv d_{euc}|_{X \times X}.$$

## Edge gluings

An edge gluing of a Euclidean polygon is a collection of isometries (w.r.t. $d_{euc}$)

$$\varphi_{2k-1} : E_{2k-1} \longrightarrow E_{2k} \qquad \varphi_{2k} := \varphi_{2k-1}^{-1}$$

btwn pairs $\{E_1, E_2\}, \{E_3, E_4\}, \ldots, \{E_{2k-1}, E_{2k}\}$ of its edges.

Namely, the edges $E_{2k-1}$ & $E_{2k}$ must have the same length. If these edges are finite, we can encode $\varphi_{2k-1}$ by decorating $E_{2k-1}$ & $E_{2k}$ with arrows. If the edges are lines or rays, we'll further need dots to determine $\varphi_{2k-1}$.

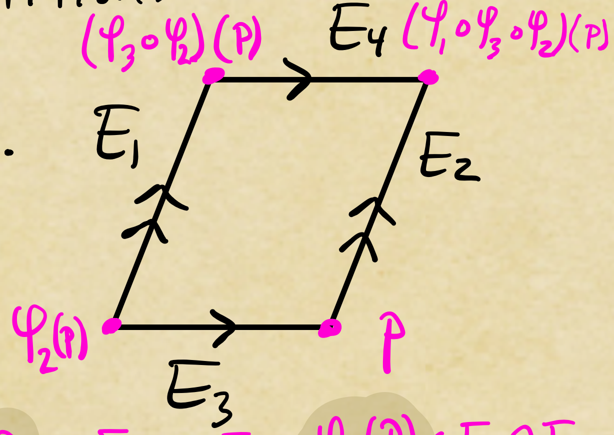
(See previous figures.)

# Edge gluings

An edge gluing of $X$ gives us a partition:

- $P \in X$ in the interior $\Rightarrow \overline{P} = \{P\}$.
- $P \in X$ in exactly one edge $E_i$

  $\Rightarrow \overline{P} = \{P, \varphi_i(P)\}$




$P \in E_2 \cap E_3$. $\varphi_2(P) \in E_1 \cap E_3$

$(\varphi_3 \circ \varphi_2)(P) \in E_1 \cap E_4$. $(\varphi_1 \circ \varphi_3 \circ \varphi_2)(P) \in E_2 \cap E_4$

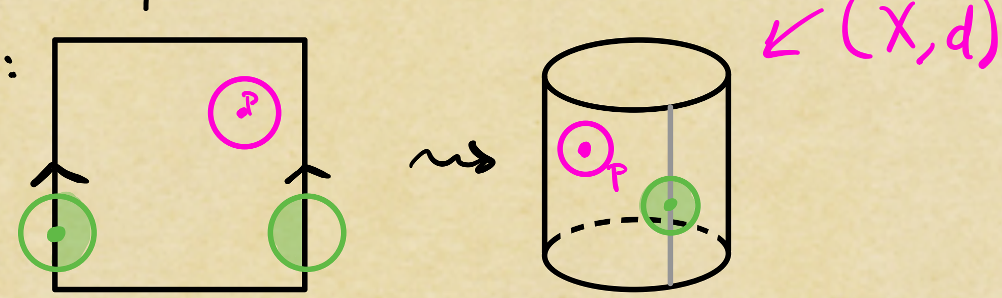
- $P \in X$ a vertex

  $\Rightarrow \overline{P} = \left\{ (\varphi_{i_k} \circ \varphi_{i_{k-1}} \circ \cdots \circ \varphi_{i_1})(P) \,\middle|\, (\varphi_{i_{j-1}} \circ \cdots \circ \varphi_{i_1})(P) \in E_{i_j}, \forall j \right\}$

<u>Thm</u>. If $\overline{X}$ is obtained from the Euclidean polygon $X$ by gluing together edge pairs by isometries, then the gluing is proper. (i.e., the semi-metric $\overline{d}_X$ <u>is a metric</u>)

# Euclidean surfaces

Our first example of a Euclidean surface was ~~the~~ a cylinder:

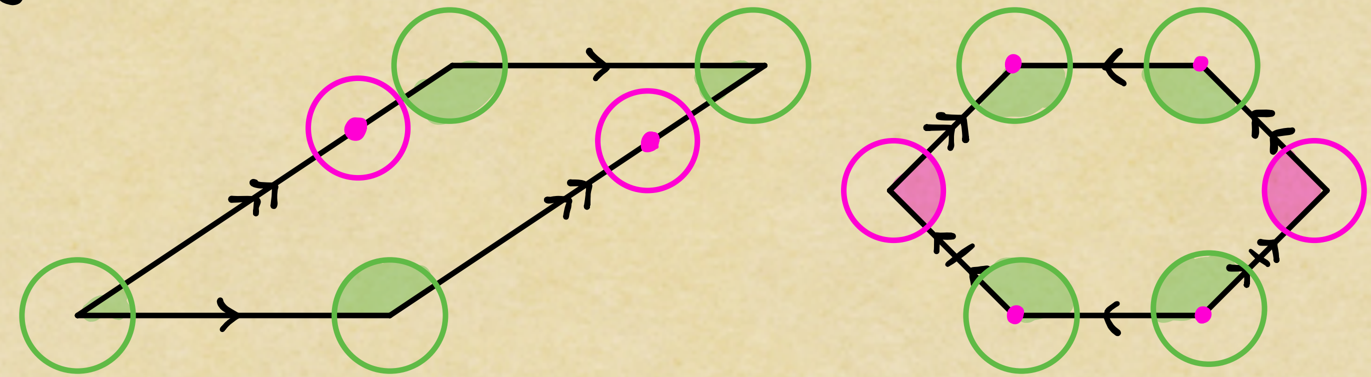



We claimed that while distances between points will change thru our gluing process, the local geometry will not. It's time to make this more precise.

**Def.** The metric space $(X,d)$ is said to be locally isometric to $(X',d')$ if, for every $P \in X$, there exist a ball $B_{d_X}(P,\varepsilon)$, a ball $B_{d_{X'}}(Q,\varepsilon)$, and an isometry between these. ⚠

# Euclidean surfaces

We call $(X,d)$ a Euclidean surface if it is locally isometric to the Euclidean plane $(\mathbb{R}^2, d_{euc})$, and say that $d$ is a Euclidean metric.



**Thm.** Let $(X,d_X)$ be a Euclidean polygon, and let $(\overline{X}, d_{\overline{X}})$ be the quotient metric space obtained from some edge gluing of $(X,d_X)$. Suppose that, for every vertex $P \in X$,

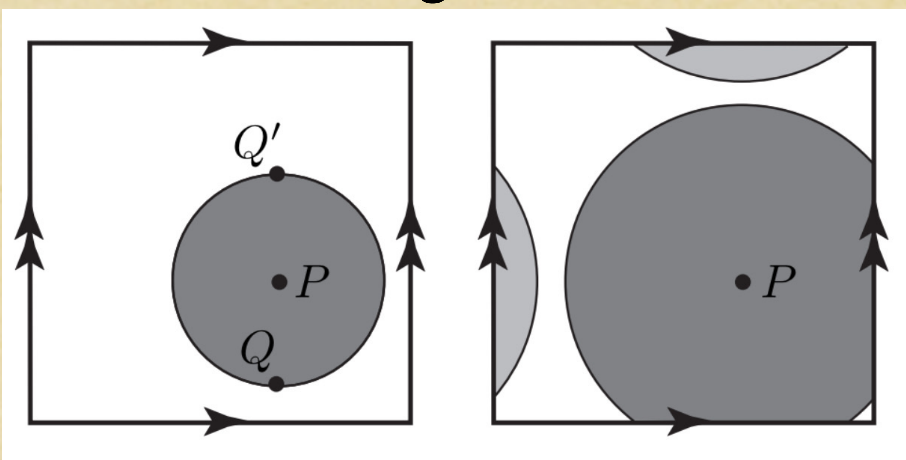
$$\sum_{Q \in \overline{P}} \measuredangle(Q) = 2\pi$$

Then $(\overline{X}, d_{\overline{X}})$ is Euclidean surface .

# Proof interlude

Our two statements about Euclidean edge gluings — properness and the criterion for surfaces — rely on two technical facts which we'll state without proof.

Lemma ① Let $(\overline{X}, \overline{d}_X)$ be obtained from the Euclidean polygon $(X, d)$ by an edge gluing. For every $\overline{P} \in \overline{X}$, $\exists\ \varepsilon_0 > 0$ s.t. for every $\varepsilon \leq \varepsilon_0$ and every $Q \in X$,



small $\varepsilon$ medium $\varepsilon$

$$\overline{Q} \in B_{\overline{d}_X}(\overline{P}, \varepsilon)$$

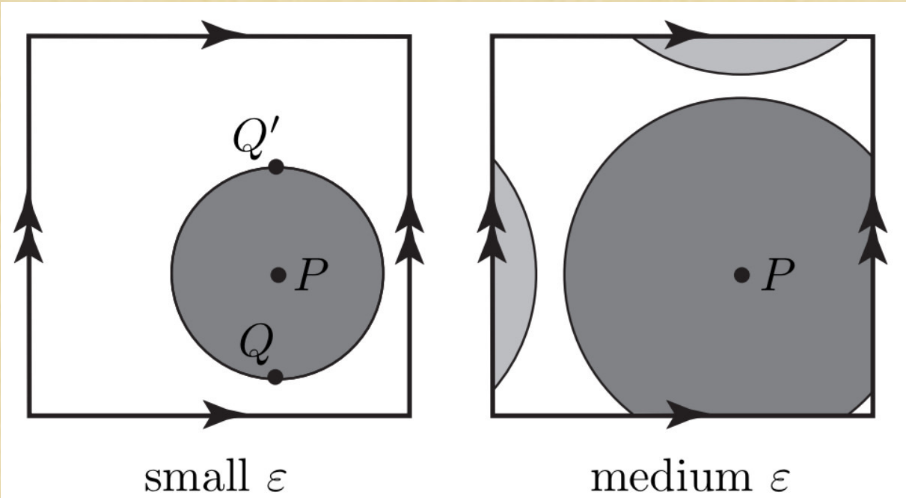
iff

$\exists\, P' \in \overline{P}$ s.t.

$$Q \in B_{d_X}(P', \varepsilon).$$

# Proof interlude

**Lemma ①** Let $(\overline{X}, \overline{d}_X)$ be obtained from the Euclidean polygon $(X, d)$ by an edge gluing. For every $\overline{P} \in \overline{X}$, $\exists\ \varepsilon_0 > 0$ s.t. for every $\varepsilon \leq \varepsilon_0$ and every $Q \in X$,




$\overline{Q} \in B_{\overline{d}_X}(\overline{P}, \varepsilon)$

iff

$\exists\, P' \in \overline{P}$ s.t.

$Q \in B_{d_X}(P', \varepsilon)$.

i.e., for suff. small $\varepsilon > 0$, $B_{\overline{d}_{\overline{X}}}(\overline{P}, \varepsilon)$ looks like a union of balls in $X$

The proof involves a lot of casework using the triangle inequality.

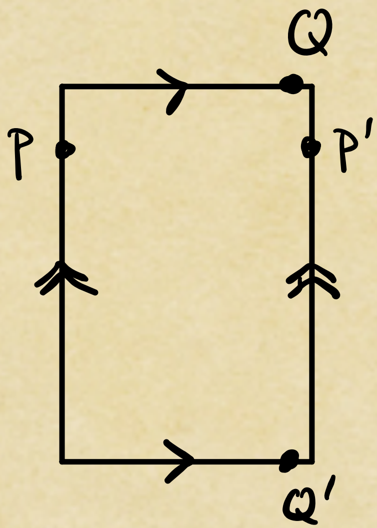
# Proof of properness

We can now verify that $\overline{d}_X$ is a metric when $(\overline{X}, \overline{d})$ is obtained from a Euclidean polygon $(X,d)$ via edge gluing.

§ $\overline{P} \neq \overline{Q}$. We NTS $\underline{\overline{d}_X(\overline{P},\overline{Q}) > 0}$.

Pick $\varepsilon_0 > 0$ for $\overline{P}$ as in Lemma 1.

$\overline{P} \neq \overline{Q} \Rightarrow \underline{\overline{P} \cap \overline{Q} = \emptyset}$ as subsets of $X$




We can choose $\varepsilon < \varepsilon_0$ s.t.

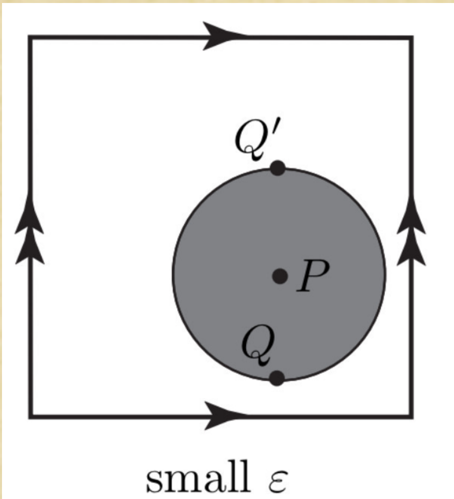
$d_X(P,Q) > \varepsilon$, $\forall P \in \overline{P}$ & $Q \in \overline{Q}$.

(Relies on $\overline{P}$ & $\overline{Q}$ being finite.)

Then $\overline{d}_X(\overline{P},\overline{Q}) \geq \varepsilon$. $\diamond$

# Proof interlude

Our other theorem says that edge gluing a Euclidean polygon yields a Euclidean surface, provided the angle sums at the vertices make sense.




The proof is similar to that of Lemma 1, in that each $\overline{P} \in \overline{X}$ needs a ball small enough for some purpose, and there's a lot of casework.

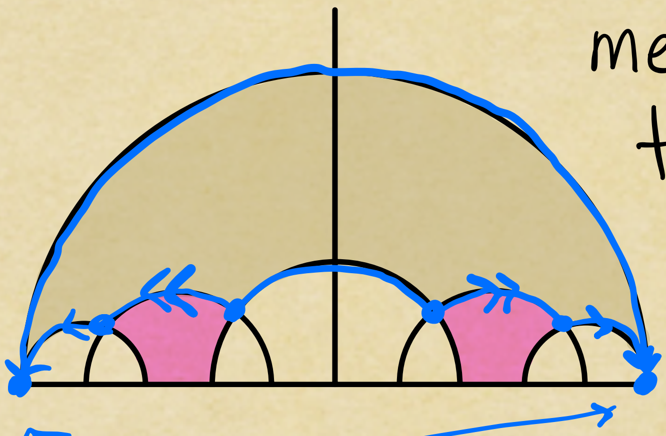
The linchpin is

**Lemma 2** Let $\varphi : g \to g'$ be an isometry btwn two geodesics in $(\mathbb{R}^2, d)$, and choose a "side" of each of $g, g'$. There is a unique isometry $\overline{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ which extends $\varphi$ and respects the sides.

# Hyperbolic surfaces

A hyperbolic polygon $X \subset (\mathbb{H}^2, d_{hyp})$ is a region whose boundary in $\mathbb{H}^2$ consists of finitely many geodesics meeting only at their endpoints. Exactly two edges meet at each vertex. We say that $X$ is unbounded if its boundary includes a nontrivial interval on the real axis.* Otherwise, $X$ is bounded.

*or it's unbounded in deuc



Gold region has trivial bdry on real axis → bounded.

Exactly as in the Euclidean case, we can define the path metric $d_X$ on $X$, as well as edge gluings, which provide pairwise isometries of the edges in $d_{hyp}$.

# Hyperbolic surfaces

Our two important theorems for edge gluings of Euclidean polygons also hold in the hyperbolic case:

Let $(\overline{X}, \overline{d}_X)$ be obtained from a hyperbolic polygon via an edge gluing by hyperbolic isometries.

**Thm.** The semi-metric $\overline{d}_X$ is in fact a metric.
i.e., the edge gluing is proper

**Thm.** Suppose that, for every vertex $P \in X$,
$$\sum_{Q \in \overline{P}} \measuredangle(Q) = 2\pi.$$
Then $(\overline{X}, d_{\overline{X}})$ is a hyperbolic surface.
i.e., $(\overline{X}, d_{\overline{X}})$ is locally isometric to $(\mathbb{H}^2, d_{hyp})$

# Spherical surfaces

A spherical polygon $X \subset (S^2, d_{sph})$ is a region whose boundary consists of finitely many geodesics meeting only at their endpoints. Exactly two edges meet at each vertex. All spherical polygons are bounded.

We define the path metric and edge gluings as usual.

Finally, let $(\overline{X}, \overline{d}_X)$ be obtained from a spherical polygon via an edge gluing by spherical isometries.

<u>Thm</u>. The semi-metric $\overline{d_X}$ is in fact a metric.

<u>Thm</u>. Suppose that, for every vertex $P \in X$,

$$\sum_{Q \in \overline{P}} \measuredangle(Q) = 2\pi.$$

Then $(\overline{X}, d_{\overline{X}})$ is a spherical surface.

i.e., locally isometric to $(S^2, d_{sph})$

## Coming up

Lots of examples!